CANONICAL CONNECTIONS

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Submitted to the Graduate Faculty of

the Kenneth P. Dietrich School of Arts and Sciences in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2014

UNIVERSITY OF PITTSBURGH

KENNETH P. DIETRICH SCHOOL OF ARTS AND SCIENCES

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February 25, 2014

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CANONICAL CONNECTIONS

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University of Pittsburgh, 2014

We study the space of canonical connections on a reductive homogeneous space. Through the investigation of lines in the space of connections invariant under parallelism, we prove that on a compact simple Lie group, bi-invariant canonical connections are exactly the bi-invariant connections that are invariant under parallelism. This motivates our definition of a family of canonical connections on Lie groups that generalizes the classical $(+)$, (0), and $(-)$ connections studied by Cartan and Schouten. We find the horizontal lift equation of each connection in this family, as well as compute the square of the corresponding Dirac operator as the element of non-commutative Weil algebra defined by Alekseev and Meinrenken.

Keywords: canonical connections, Cartan-Schouten connections, Dirac operators.

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I would like to thank everyone who has helped me during my Ph.D. studies.

1.0 INTRODUCTION

It is known that if M is a manifold with a linear connection that is invariant under its own parallelism (or, equivalently, whose torsion and curvature are parallel), then M is an analytic manifold and the connection is itself analytic. If, in addition, M is connected, simply connected and the connection is complete then $\mathrm{Aff}(M)$, the group of affine transformations, is a Lie group that acts transitively on M, endowing it with a structure of homogeneous space.

This situation is familiar in the context of affine locally symmetric spaces which are manifolds with a linear, torsion-free connection whose curvature is parallel. If the affine symmetric space is simply connected and the connection is complete then it is an affine symmetric space.

Manifolds with linear connections that are invariant under parallelism are therefore a natural extension of the notion of affine locally symmetric space. Since under some conditions (M connected, simply connected, complete connection) such a manifold is a homogeneous space and the connection is invariant, a more amenable context is that of homogeneous spaces K/H and K-invariant linear connections that are invariant under parallelism. The set of K-invariant linear connections on K/H is parametrized by a space of linear maps, according to a theorem of Wang (Theorem 2.83). In this parametrization, the connections associated with $\mathrm{Ad}(H)$-stable complements of $\mathrm{Lie}(H)$ in $\mathrm{Lie}(K)$ are called *canonical connections*. Canonical connections are necessarily complete and invariant under parallelism. This class of connections is the subject of this dissertation. Next we will give a brief account of our main results.

Let K/H be a homogeneous space. In Chapter 3 we consider the question of whether its subset of connections invariant under parallelism contains lines, and we derive some algebraic necessary and sufficient conditions for this to happen (Theorem 3.23, Theorem 3.24, and Theorem 3.25) and necessary and sufficient conditions for a certain distinguished line (the bracket connections) to consists of connections invariant under parallelism (Theorem 3.28).

In Chapter 4 we study in detail the homogeneous space $S \times S/\Delta(S \times S)$ for S a connected compact Lie group. In this case all self parallel invariant connections form a line whose elements are called *Cartan-Schouten connections*. We also record various invariants of these connections: torsion, curvature, holonomy Lie algebra. We also explicitly work out the horizontal lift equation, which is the ODE satisfied by the horizontal lift of a curve on S with respect to such a connection (Theorem 4.22).

Dirac operators were introduced into the representation theory of a real reductive group by Parthasarathy [21] with the purpose of constructing discrete series representations. Kostant [14] introduced a Dirac operator that contains terms of degree 3 (which he called a cubic Dirac operator) associated to a Lie algebra, a quadratic subalgebra and an orthogonal complement. He used the cubic Dirac operator to obtain an algebraic version of the Borel-Weil construction that does not require the existence of complex structures, a generalization of the Weyl character formula, and to discover multiplets of representations [10]. Such an operator is a geometric Dirac operator on a homogeneous space and in the case of $S \times S/\Delta(S \times S)$ it corresponds to a Cartan-Schouten connection. The geometric Dirac operators associated to Cartan-Schouten connections were studied by Slebarski [22] who obtained a formula for their square. The relationship with the cubic Dirac operator was pointed out in [1] and [9].

In Chapter 5 we study the twisted Dirac operator associated to a Cartan-Schouten connection. These are elements of the non-commutative Weil algebra defined in [2]. In Theorem 5.46 we obtain a formula for their square in terms of the Casimir operator on S, similar to the one obtained by Slebarski in the geometric context.

2.0 PRELIMINARIES

For the motivation and expository purposes we present a succinct overview of the theory of connections in principal bundles and related concepts. Our main reference is Kobayashi and Nomizu [13].

2.1 PRINCIPAL BUNDLES

We fix the following notation that will be used henceforth. Let M be an n-dimensional differentiable manifold, G a Lie group, and $\mathfrak{g}$ the Lie algebra of G. Let $\mathrm{Ad} : G \to GL(\mathfrak{g})$ be the adjoint representation of G and $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ the adjoint representation of $\mathfrak{g}$. Let $\exp_G : \mathfrak{g} \to G$ denote the exponential map of G.

Definition 2.1. Let E, B, and F be topological spaces. A continuous surjection $\pi : E \to B$ is said to be *locally trivial with standard fibre* F if for every $x \in E$, there is an open neighborhood $U \subset B$ of $\pi(x)$ and a homeomorphism $\phi : \pi^{-1}U \to U \times F$, such that $\pi = p_1 \circ \phi$ where $p_1 : U \times F \to U$ is the natural projection.

Definition 2.2. A *fibre bundle* consists of the data (E, B, π, F), where E, B, and F are smooth manifolds and $\pi : E \to B$ is a locally trivial differentiable surjection with standard fibre F. The space B is called the *base space* of the bundle, E the *total space*, and F the *standard fibre*.

Definition 2.3. A (*differentiable*) *principal* G*-bundle* (P, G, R) over M consists of a manifold P and a right action $R : P \times G \to P, (u, a) \mapsto u \cdot a$ such that

(i) R is a free action,

(ii) M is the quotient space of G-orbits in P, and

(iii) $\pi : P \to M$ is locally-trivial with standard fibre G.

Notation.

(1) A principal fibre bundle will be denoted by $P(M, G)$, or by $\pi : P \to M$, or, if the other ingredients are clear from the context, simply by P. For $u \in P$, the orbit $u \cdot G$ is denoted by P_u.

(2) For $x \in G$, denote by $l(x) : G \to G$, $l(x)y = xy$ the left multiplication, and by $r(x) : G \to G$, $r(x)y = yx$ the right multiplication, by x.

(3) For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of M. If $u \in \pi^{-1}(x)$, then it follows from Definition 2.3 that $\pi^{-1}(x) = P_u$. For each $a \in G$ and $u \in P$, the diffeomorphism of P_u induced by the G action is denoted, for each $v \in P_u$, by

$$R_a : P_u \to P_u, v \mapsto v \cdot a.$$

Example 2.4. A *linear frame* u at a point $x \in M$ is an ordered basis of the tangent space T_xM. Let LM be the differentiable manifold of all linear frames at all points of M and let $\pi : LM \to M$ map a linear frame u at x into x. The general linear group $GL_n\mathbb{R}$ acts on LM on the right as follows. If $a = (a^i_j) \in GL_n\mathbb{R}$ and $u = \{X_1, \cdots, X_n\}$ is a linear frame at x, then $u \cdot a$ is, by definition, the linear frame $\{Y_1, \cdots, Y_n\}$ at x defined by $Y_i = \sum_j a^j_i X_j$. It is clear that $GL_n\mathbb{R}$ acts freely on LM and that $\pi(u) = \pi(v)$ if and only if $v = u \cdot a$ for some $a \in GL_n\mathbb{R}$. Then, LM is a principal $GL_n\mathbb{R}$-bundle, called the *linear frame bundle* over M.

Example 2.5. Let S be a Lie group and $\mathfrak{s}$ its Lie algebra. To describe the bundle of linear frames LS, we fix a frame ν on $\mathfrak{s}$. Denote $u_e = (e, \nu)$, where $e \in S$ is the identity element. Consider the section $l^\nu(x) = (x, l(x)_{*,e}\nu)$ in LS. It follows that l^ν gives rise, via the correspondence $(x, l(x)_{*,e}\nu \cdot a) \leftrightarrow (x, a)$, to a global trivialization

$$LS \simeq S \times GL(\mathfrak{s}).$$

2.1.1 Fundamental vector fields.

Let P be a principal G-bundle over M.

Definition 2.6. For each $a \in G$ and $u \in P$, let

$$\sigma_u : G \simeq P_u,\ a \mapsto u \cdot a.$$

We define the *fundamental vector field* in P induced by $X \in \mathfrak{g}$ by

$$(\sigma_u)_{*,e}X = \frac{d}{dt}|_{t=0} u \cdot \exp_G tX,$$

where $e \in G$ is the identity element.

Notation. This vector $(\sigma_u)_{*,e}X$ is typically denoted by $({}^{\sigma}X)_u$, and the corresponding map is denoted by ${}^{\sigma} : \mathfrak{g} \to \mathfrak{X}(P), X \mapsto {}^{\sigma}X$.

Proposition 2.7. [13, vol. 1, I, Proposition 4.1, Proposition 5.1] *Let P be a principal G-bundle and let $u \in P$. The induced map $(\sigma_u)_{*,e} : \mathfrak{g} \to T_uP_u$ is a linear isomorphism that makes the diagram*

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "(\sigma_u)_{*,e}"] \arrow[d, "\mathrm{Ad}(a^{-1})"'] & T_uP_u \arrow[d, "(R_a)_{*,u}"] \\
\mathfrak{g} \arrow[r, "(\sigma_{u\cdot a})_{*,e}"] & T_{u\cdot a}P_u
\end{tikzcd}$$

commutative for each $a \in G$.

2.1.2 Associated vector bundles.

Let $P(M, G)$ be a principal G-bundle and $\rho : G \to GL(V)$ a Lie group representation. On $P \times V$, we let G act on the right:

$$P \times V \times G \to P \times V,\ (u, \xi, a) \mapsto (u \cdot a, \rho(a)^{-1}\xi) \in P \times V.$$

The action is free, and the space of G-orbits is denoted by $E = P \times_\rho V$.

Definition 2.8. The map $P \times V \to M$, $(u, \xi) \mapsto \pi(u)$ induces a map $\pi_E : E \to M$. E admits a natural structure of differentiable manifold and $\pi_E : E \to M$ becomes a vector bundle with standard fibre V. $E = P \times_\rho V$ is called the *vector bundle associated to P and ρ.*

Notation. For each $u \in P$ and each $\xi \in V$, we denote by $u\xi$ the image of $(u, \xi) \in P \times V$ through the canonical projection $P \times V \to E$.

Proposition 2.9. [13, vol. 1, I, Proposition 5.4] *Let $E = P \times_\rho V$ be the vector bundle associated to P and ρ. There exists a linear isomorphism $u : V \simeq E_{\pi(u)}$, $\xi \mapsto u\xi$ for each $u \in P$, which satisfies the transformation rule*

$$(u \cdot a)\xi = u(\rho(a)\xi)$$

for each $a \in G$ and $\xi \in V$.

Example 2.10. Let LM be the linear frame bundle over a manifold M. Let

$$\mathrm{id} : GL(\mathbb{R}^n) \to GL(\mathbb{R}^n)$$

denote the standard representation and id^* the contragradient representation $\mathrm{id}^*(a) = (a^{-1})^t$. Then $TM = LM \times_{\mathrm{id}} \mathbb{R}^n$ and $T^*M = LM \times_{\mathrm{id}^*} (\mathbb{R}^n)^*$, where TM and T^*M are the tangent and cotangent bundles of M, respectively. Similarly, $\Lambda^k TM = LM \times_{\Lambda^k \mathrm{id}} \Lambda^k \mathbb{R}^n$, $\Lambda^k T^*M = LM \times_{\Lambda^k \mathrm{id}^*} (\Lambda^k \mathbb{R}^n)^*$, and $\bigotimes_s^r TM = LM \times_{(\otimes^r \mathrm{id}) \otimes (\otimes^s \mathrm{id}^*)} \bigotimes_s^r \mathbb{R}^n$, where $\Lambda^k \mathrm{id}$, $\Lambda^k \mathrm{id}^*$, and $(\otimes^r \mathrm{id}) \otimes (\otimes^s \mathrm{id}^*)$ are the induced exterior power and tensor product representations, and $\bigotimes_s^r \mathbb{R} = \mathbb{R}^{\otimes^r} \bigotimes (\mathbb{R}^*)^{\otimes^s}$.

2.2 CONNECTIONS

The geometric notion of connection originates with T. Levi-Civita's parallelism [18] and was later generalized to the notion of connection of differentiable fibre bundles by C. Ehresmann [8]. Notions such as linear connections can be described in terms of bundles constructed from the tangent bundles of differentiable manifolds.

Definition 2.11. Let $\pi : P \to M$ be an (infinitely) differentiable principal G-bundle. The tangent vector space T_uP at each point $u \in P$ is mapped by $\pi_{*,u}$ onto the tangent vector space T_xM at the point $x = \pi(u) \in M$. The kernel $\mathfrak{Y}(u)$ of the map $\pi_{*,u} : T_uP \to T_xM$ is called the *vertical subspace* of T_uP at u, and each vector in $\mathfrak{Y}(u)$ is called a *vertical vector*.

Definition 2.12. A *connection* in the principal G bundle P is a differentiable distribution $\Gamma = \{\Gamma(u)\}_{u \in P}$ of subspaces $\Gamma(u) \subset T_uP$ such that

(i) $T_uP = \Gamma(u) \bigoplus \mathfrak{Y}(u)$, and

(ii) Γ is invariant under G, i.e. $(R_a)_{*,u}\Gamma(u) = \Gamma(u \cdot a)$ for all $a \in G$.

$\Gamma(u)$ is said to be the *horizontal subspace* in T_uP determined by Γ, and a vector in $\Gamma(u)$ is said to be *horizontal* with respect to Γ.

Definition 2.13. A *linear connection* is a connection in a linear frame bundle.

Notation. The collection of all connections in P will be denoted by $\mathrm{Conn}(P; M)$. If $P = LM$ is understood, then the collection of all linear connections $\mathrm{Conn}(LM; M)$ will be denoted by $\mathrm{Conn}(M)$ for simplicity.

Now suppose that X is an arbitrary vector field on P. The direct sum decomposition $T_uP = \Gamma(u) \bigoplus \mathfrak{Y}(u)$ implies that the value X_u of X at each point $u \in P$ can be expressed uniquely as $X_u = X_u^{=} + X_u^{\perp}$, where $X_u^{=} \in \Gamma(u)$ and $X_u^{\perp} \in \mathfrak{Y}(u)$. The vector fields $X^{\perp}$ and $X^{=}$ are called the *vertical* and *horizontal component* of X, respectively.

Remark 2.14.

(1) The kernel $\mathfrak{Y}(u)$ equals T_uP_u, the tangent space at u to the fibre P_u.

(2) The differentiability of the distribution Γ implies that for any differentiable vector field, its horizontal and vertical components are also differentiable vector fields.

2.2.1 Connection forms.

Let P be a principal G-bundle over M.

Definition 2.15. For a given connection Γ in P, we define its *connection form*

$$\omega_\Gamma : TP \to \mathfrak{g},\ \omega_\Gamma(u)X = (\sigma_u^{-1})_{*,u}X_u^{\perp},$$

where $u \in P$, $X^{\perp}$ is the vertical component of $X \in T_uP$ with respect to Γ.

Notation. When there is no danger of confusion, we shall denote ω_Γ by ω.

Remark 2.16.

(1) It follows from the commutative diagram in Proposition 2.7 that $R_a^*\omega = \mathrm{Ad}(a^{-1})\omega$ for all $a \in G$.

(2) $\omega(u)|_{T_u P_u} : T_u P_u \to \mathfrak{g}$ is the inverse of $(\sigma_u)_{*,e}$.

(3) $\ker \omega(u) = \Gamma(u)$.

Definition 2.17. We define a class $\mathfrak{C}(M;P) \subset \mathfrak{g} \bigotimes T^*P$ of *connection-type forms* in P as follows: $\omega \in \mathfrak{C}(M;P)$ if and only if

(i) $\omega(u)({}^\sigma X)_u = X$ for all $X \in \mathfrak{g}$, $u \in P$, and

(ii) $R_a^* \omega = \mathrm{Ad}(a^{-1})\omega$ for all $a \in G$.

Proposition 2.18. *The map $\omega : \Gamma \mapsto \omega_\Gamma$ gives rise to a bijective correspondence between* $\mathrm{Conn}(M;P)$ *and* $\mathfrak{C}(P;M)$.

Proof. We know from Remark 2.16(1) that the connection form ω_Γ must satisfy $R_a^* \omega_\Gamma = \mathrm{Ad}(a^{-1})\omega_\Gamma$ for all $a \in G$. Conversely, given an $\omega \in \mathfrak{C}(M;P)$, we can construct a connection in P by defining $\Gamma(u) = \ker \omega(u)$ at each $u \in P$, as Remark 2.16(3) indicates. □

If Γ is a connection in P, then there exists a projection $h_\Gamma : T_u P \to \Gamma(u)$ relative to the decomposition $T_u P = \Gamma(u) \bigoplus \mathfrak{Y}(u)$. The form $D\phi = (d\phi) \circ h_\Gamma$ is called the *exterior covariant derivative* of the pseudo-tensorial form ϕ [13, vol. 1, p. 75], and D is called the *exterior covariant differentiation* determined the by connection Γ.

Definition 2.19. For an arbitrary connection form $\omega \in \mathrm{Conn}(P;M)$,

$$\Omega = D\omega$$

is called the *curvature form* of ω. Let $E = P \times_\rho V$ for some Lie group representation $\rho : G \to GL(V)$. We define the *canonical form* θ of P as the V-valued form $\theta X = u^{-1}(\pi_* X)$ for $X \in T_u P$, where $u \in P$ is regarded as a linear isomorphism $V \simeq E_x$ with $x = \pi(u)$. The *torsion form* Θ of the connection Γ in P is then defined by

$$\Theta = D\theta.$$

The forms ω, Ω, and Θ are related by the structural equations of E. Cartan:

Theorem 2.20. [13, vol. 1, III, Theorem 2.4] *Let ω, Θ, and Ω be the connection form, the torsion form, and the curvature form of a connection in P. Then*

$$\Omega = d\omega + \frac{1}{2}\omega \wedge \omega, \quad \Theta = d\theta + \omega \wedge \theta.$$

2.2.2 Parallel transport.

Let $\pi : P \to M$ be a principal G-bundle equipped with a connection Γ, let X be a vector field on M, $u \in P$ and $x = \pi(u) \in M$. Since π induces an isomorphism $\pi_{*,u} : \Gamma(u) \simeq T_xM$, there is a unique vector field X^* on P such that $X^*_u \in \Gamma(u)$ and $\pi_* X^* = X$.

Definition 2.21. X^* is called the *horizontal lift* of X. A *horizontal curve* in P with respect to Γ is a piecewise differentiable curve of class C^1 whose tangent vectors are all horizontal with respect to Γ. Let c be a piecewise differentiable curve of class C^1 in M. A *horizontal lift* of c is a horizontal curve c^* in P such that $\pi \circ c^* = c$.

Now we define the notion of parallel transport with respect to a connection. It requires the following fundamental theorem:

Theorem 2.22. [13, vol. 1, II, Proposition 3.1] *Let $\pi : P \to M$ be a principal G-bundle equipped with a connection Γ. Let c be a curve of class C^1 in M and t_0 be a point in the domain of c. For an arbitrary point $u_0 \in \pi^{-1}(c(t_0))$, there exists a unique horizontal lift c^* in P of c such that $c^*(t_0) = u_0$.*

Definition 2.23. c^* is called *the horizontal lift* of c through u_0 with respect to Γ.

Let c be a differentiable curve of class C^1 on M. We define a map of the fibre $\pi^{-1}(c(a))$ onto the fibre $\pi^{-1}(c(b))$ as follows. Let u_a be an arbitrary point of $\pi^{-1}(c(a))$. The horizontal lift c^* of c through u_a with respect to Γ goes through a point u_b such that $\pi(u_b) = c(b)$. By varying u_a in the fibre $\pi^{-1}(c(a))$, we obtain a map from the fibre $\pi^{-1}(c(a))$ to the fibre $\pi^{-1}(c(b))$ which maps u_a into u_b.

Definition 2.24. The map

$$\tau_c|_b^a : \pi^{-1}(c(a)) \simeq \pi^{-1}(c(b))$$

described above is called the *parallel transport* along c.

Notation. If $c : [a, b] \to M$, then the inverse curve c^{-1} denotes, as usual, $t \mapsto c(a + b - t)$ for $a \le t \le b$. If the end point of c meets the starting point of another curve d, then their concatenation (c followed by d) curve is denoted by $c * d$.

Proposition 2.25. [13, vol. 1, II, Proposition 3.2, Proposition 3.3]

(*a*) *The parallel transport along any curve c commutes with the action of G on P : for all $a \in G$,*

$$\tau_c \circ R_a = R_a \circ \tau_c. \tag{2.2.1}$$

(*b*) *If c is a piecewise differentiable curve of class C^1 on M, then $\tau_{c^{-1}} = \tau_c^{-1}$. If c is a curve from x to y in M and d is a curve from y to z in M, then $\tau_{c*d} = \tau_d \circ \tau_c$.*

We now discuss the notion of parallel transport on the associated vector bundle $E = P \times_\rho V$ of a principal G-bundle P. Let Γ be a connection in P, and let $w \in E$. We shall define the notion of horizontal subspace and vertical subspace in $T_w E$. First, we choose a point $(u, \xi) \in P \times V$ which is mapped into w under the canonical projection $P \times V \to E$. We fix this $\xi \in V$ and consider the map

$$\xi : P \to E, v \mapsto v\xi.$$

Definition 2.26. The *horizontal subspace* $\Gamma(w, E)$ is defined to be the image of the horizontal subspace $\Gamma(u) \subset T_u P$ under the map $\xi : v \mapsto v\xi$. The *vertical subspace* $\mathfrak{Y}(w, E)$ is defined as the tangent space to the fibre of E at w. It is easy to verify that $\Gamma(w, E)$ is independent of the choice of $(u, \xi) \in P \times V$, and that $T_w E = \Gamma(w, E) \bigoplus \mathfrak{Y}(w, E)$. A curve in E is *horizontal* if its tangent vectors are horizontal at each point in E. Let c be a curve in M. A *horizontal lift* c^* of c is a horizontal curve in E such that $\pi_E \circ c^* = c$.

Proposition 2.27. *Let $c : [0, 1] \to M$ be a curve, and choose a point $w_0 \in E$ such that $\pi_E(w_0) = x_0$. Then there exists a unique horizontal lift c^* starting from w_0.*

Proof. To prove the existence of c^*, we choose a point (u_0, ξ) in $P \times V$ such that $u_0\xi = w_0$. Let u be the horizontal lift of c in P through u_0. Then $u\xi$ is a horizontal lift of c in E through w_0. The uniqueness of c^* follows from the uniqueness of solution of system of ordinary linear differential equations, just as in the case of the horizontal lift in a principal G-bundle. $\square$

Now, Γ defines the notion of parallel transport of fibres of $E = P \times_\rho V$, in an analogous way as in the case of principal G-bundles. More precisely, let c be a differentiable curve of class C^1 on M, and let w_a be an arbitrary point of $E_{c(a)}$. The horizontal lift c^* of c through w_a with respect to Γ goes through a point w_b such that $\pi_E(u_b) = c(b)$. By varying w_a in the fibre $E_{c(a)}$, we obtain a map from the fibre $E_{c(a)}$ to the fibre $E_{c(b)}$ which maps w_a into w_b.

Definition 2.28. The map

$$\tau_c^E|_b^a : E_{c(a)} \simeq E_{c(b)}$$

described above is called the *parallel transport in* $E = P \times_\rho V$ along c.

Notation. This will be also denoted by τ_c henceforth, as no misunderstanding of the correct context can happen in each of the situations we shall encounter later on.

2.2.3 Holonomy groups.

Using the notion of parallel transport, we now define the holonomy group of a given connection Γ in a principal G-bundle P. For simplicity we shall mean by a *curve* a piecewise differentiable curve of class C^k for some fixed $1 \leq k \leq \infty$.

For each fixed point $x \in M$, let $L(x)$ be the loop space at x, that is, the set of all closed curves starting and ending at x. If $c, d \in L(x)$, then the concatenation curve $c * d$ is also an element of $L(x)$. Proposition 2.25(a) shows that τ_c is an isomorphism of the fibre $\pi^{-1}(x)$ onto itself. The set $\Phi(x)$ of all such isomorphisms of $\pi^{-1}(x)$ onto itself forms a group by Proposition 2.25(b). Let $L^0(x) \subset L(x)$ be the subset of loops homotopic to the constant loop.

Definition 2.29. This group $\Phi(x)$ of isomorphisms is called the *holonomy group of* Γ *at* x. This subgroup $\Phi^0(x)$ of $\Phi(x)$ consisting of the elements arising from $L^0(x)$ is called the *restricted holonomy group of* Γ *at* x. These two groups may be realized as subgroups of the structure group G in the following way. Let $x \in M$ and fix $u \in \pi^{-1}(x)$. Each $c \in L(x)$ determines an element $a_c \in G$ such that $\tau_c(u) = u \cdot a_c$. If a loop $d \in L(x)$ determines $a_d \in G$, then $d * c$ determines $a_d a_c$ because

$$(\tau_d \circ \tau_c)(u) = \tau_d(u \cdot a_c) = \tau_d(u)a_c = u \cdot a_d a_c$$

by (2.2.1). Therefore,

$$\{a_c | c \in L(x)\} =: \Phi(u)$$

forms a subgroup of G. The subgroup $\Phi(u)$ is called the *holonomy group* of Γ with reference point $u \in P$. The *restricted holonomy group* $\Phi^0(u)$ of Γ with reference point u can be defined accordingly.

The holonomy groups do not depend on the choice of curve regularity due to the following nontrivial result:

Theorem 2.30. [13, vol. 1, II, Theorem 7.2, Corollary 7.3] *Let $\Phi_k(u)$ denote the holonomy group obtained from piecewise differentiable curves of class C^k. Then $\Phi_k(u)$ coincide, and so do $\Phi_k^0(u)$, for all $1 \leq k \leq \infty$.*

Another way of defining $\Phi(u)$ is the following. When two points u and $v \in P$ can be joined by a horizontal curve, we write $u \sim v$ (this is clearly an equivalence relation). Then

$$\Phi(u) = \{a \in G | u \sim u \cdot a\}.$$

Using that $u \sim v$ implies $u \cdot a \sim v \cdot a$ for any $u, v \in P$ and $a \in G$, it is easy to verify once more that this $\Phi(u)$ forms a subgroup of G.

Proposition 2.31. [13, vol. 1, II, Proposition 4.1, Theorem 4.2] *Suppose P is a principal G-bundle with a connection. Let $u \in P$ and $a \in G$.*

(a) $\Phi(u \cdot a) = a^{-1}\Phi(u)a$. Similarly, $\Phi^0(u \cdot a) = a^{-1}\Phi^0(u)a$.

(b) If $u \sim v$ in P, then $\Phi(u) = \Phi(v)$ and $\Phi^0(u) = \Phi^0(v)$.

If M is assumed to be connected and paracompact, then

(c) $\Phi(u)$ is a Lie subgroup of G, and

(d) $\Phi^0(u)$ is the identity component of G, and $\Phi(u)/\Phi^0(u)$ is countable.

Definition 2.32. Let P be a principal G-bundle equipped with a connection. The set $P(u)$ of points in P that can be joined to $u \in P$ by a horizontal curve is called the *holonomy bundle* through u.

The following result is due to W. Ambrose and I. Singer [3].

Theorem 2.33. [13, vol. 1, II, Theorem 8.1] *Let P be a principal G-bundle, where M is connected and paracompact. Let Γ be a connection in P, Ω its curvature form, $\Phi(u)$*

the holonomy group with reference point $u \in P$, and $P(u, \Gamma)$ the holonomy bundle of Γ through u. Then the Lie algebra of $\Phi(u)$ is given by

$$\mathrm{Lie}(\Phi(u)) = \mathrm{span}_{\mathbb{R}}\{\Omega(v)(X, Y) | v \in P(u, \Gamma) \text{ and } X, Y \in \Gamma(v)\}.$$

Definition 2.34. A pair

$$(f, f') : P'(M', G') \to P(M, G)$$

is said to be a *morphism of principal bundles* if $f : P' \to P$ is a differentiable map and $f' : G' \to G$ is a Lie group morphism such that $f(u' \cdot a') = f(u') \cdot f'(a')$ for all $u' \in P'$ and $a' \in G'$. The principal bundle morphism is called an *embedding* if $f : P' \to P$ is an embedding and $f' : G' \to G$ is a monomorphism. If $f : P' \to P$ is an embedding, then the induced mapping $\bar{f} : M' \to M$ is also an embedding. By identifying P' with $f(P')$, G' with $f'(G')$, and M' with $\bar{f}(M')$, we say that $P'(M', G')$ is a *principal subbundle*, or simply a *subbundle*, of $P(M, G)$.

Definition 2.35. If, moreover, $M' = M$ and the induced mapping $\bar{f} : M' \to M$ is the identity transformation of M, then the bundle morphism (f, f') is called a *reduction* of the structure group G of $P(M, G)$ to G', in which case the subbundle $P'(M, G')$ is called a *reduced subbundle* of $P(M, G)$.

Example 2.36. Suppose M is a Riemannian manifold. The *orthonormal frame bundle* OM of M is the set of all orthonormal frames, relative to the given Riemannian metric, at each point $x \in M$. The orthonormal frame bundle OM is a principal O_n-bundle over M, as the orthogonal group O_n acts freely and transitively on the right on the set of all orthonormal frames. If M is orientable, then one can define the *oriented orthonormal frame bundle* O^+M of all positively-oriented orthonormal frames. This is a principal SO_n-bundle over M. OM and O^+M are reduced subbundles of the linear frame bundle LM, with reductions of structure groups to O_n and SO_n, respectively.

We discuss another important example of reduced subbundle in $P(M, G)$.

Proposition 2.37. [13, vol. 1, II, Proposition 6.1, Proposition 6.2] *Let $(f, f') : P'(M', G') \to P(M, G)$ be a bundle homomorphism such that the induced mapping $\bar{f} : M' \to M$ is a diffeomorphism of M' onto M.*

(a) If Γ' is a connection in P', then there is a unique connection Γ in P such that the horizontal subspaces of Γ' are mapped into horizontal subspaces of Γ by f. If the connection form of Γ' is ω', then the connection form of Γ is $f'_*\omega'$.

(b) If we assume, in addition to the above conditions, that $f'_* : \mathfrak{g}' \to \mathfrak{g}$ is an isomorphism, then for each connection Γ in $P(M, G)$, there is a unique connection Γ' in $P'(M', G')$ whose horizontal subspaces are mapped into those of Γ by f.

Definition 2.38. Under the hypothesis of Proposition 2.37 (a), we say that f *maps the connection* Γ' *into the connection* Γ and write $f_*(\Gamma') = \Gamma$. If $P'(M', G')$ is a reduced subbundle of $P(M, G)$ where $M' = M$ and $\bar{f} : M' \to M$ is the identity transformation, we say that Γ in P is *reducible* to $\Gamma' = f_*(\Gamma)$ in P'.

Proposition 2.39. [13, vol. 1, II, Theorem 7.1] *Let $P(M, G)$ be a principal G-bundle with a connection Γ, where M is connected and paracompact. Let $u_0 \in P$. Then*

(a) $P(u_0)$ is a reduced subbundle of P with structure group $\Phi(u_0)$.

(b) The connection Γ is reducible to a connection in $P(u_0)$.

2.2.4 Covariant derivatives.

Let $\pi : P \to M$ be a principal G-bundle and

$$\rho : G \to GL(\mathbb{R}^n)$$

a group representation. (Recall that $n = \dim M$.) We consider $E = P \times_\rho \mathbb{R}^n$, the associated vector space bundle with standard fibre $\mathbb{R}^n$. E is hence a real vector bundle over M, and each fibre $\pi_E^{-1}(x)$ over $x \in M$ of E has a structure of a vector space such that for every $u \in P$ with $\pi(u) = x$, the corresponding map

$$u : \mathbb{R}^n \to \pi_E^{-1}(x)$$

is a linear isomorphism. We fix a connection Γ in P and let ϕ be a section of E defined on a curve $c : [a, b] \to M$.

Definition 2.40. The *covariant derivative* of ϕ in the direction of $\dot{c}(t)$ is defined by

$$\nabla_{\dot{c}(t)}\phi = \lim_{h \to 0} \frac{1}{h}(\tau_c|_t^{t+h}\phi(c(t+h)) - \phi(c(t))) \in \pi_E^{-1}(c(t)).$$

Next, let $X \in T_xM$, ϕ a section of E defined in a neighborhood of x, and let $c : (-\epsilon, \epsilon) \to M$ be a curve such that $X = \dot{c}(0)$. The *covariant derivative* $\nabla_X \phi$ *of* ϕ *in the direction of* X is defined by $\nabla_X \phi = \nabla_{\dot{c}(0)}\phi$. In general, the *covariant derivative* $\nabla_X \phi$ *of* ϕ *in the direction of a vector field* X in M is defined by $(\nabla_X \phi)_x = \nabla_{X_x}\phi$ for each $x \in M$.

Now suppose that a section ϕ of E is defined on an open subset U of M. We associate with ϕ a function

$$f^\phi : \pi^{-1}U \to \mathbb{R}^n, v \mapsto v^{-1}(\phi(\pi(v))$$

for $v \in \pi^{-1}U$. Given $X \in T_xM$, let $X_u^* \in T_uP$ be the horizontal lift of X through u. Since f^ϕ is an $\mathbb{R}^n$-valued function, $u(X_u^* f^\phi)$ is an element of the fibre $E_{\pi(x)}$. If $v \in P_u$, then there exists a unique $a \in G$ where $v = u \cdot a$. It is clear from Definition 2.12(ii) that horizontal lifts are right G-invariant, namely

$$X_v^* = X_{u \cdot a}^* = (R_a)_{*,u} X_u^*.$$

Proposition 2.9 then gives $v(X_v^* f^\phi) = u(\rho(a)(R_a)_{*,u} X_u^* f^\phi)$. But

$$X_u^*(f^\phi \circ R_a) = X_u^*(\rho(a)^{-1} \circ f^\phi) = \rho(a^{-1}) X_u^* f^\phi;$$

this shows that $v(X_v^* f^\phi) = u(X_u^* f^\phi)$.

Proposition 2.41. [13, vol. 1, III, Lemma of Proposition 1.1] *Let* ϕ *be a section in* E *and let* X *be a vector field in* M*. Then*

$$\nabla_X \phi = u(X^* f^\phi).$$

Example 2.42. As Example 2.10 indicates, the principal $GL(\mathbb{R}^n)$-bundle LM of linear frames in M determines the tangent bundle $TM = LM \times_{\mathrm{id}} \mathbb{R}^n$. Hence every connection Γ in LM determines a unique covariant derivative ∇ in TM. In this case, Proposition 2.41 reads $\nabla_X Y = u(X^* u^{-1} Y_{\pi(u)})$, where Y is a vector field in M.

Proposition 2.43. [13, vol. 1, III, Proposition 2.8, Proposition 7.5] *Let* ∇ *be the covariant derivative determined by a linear connection on* M*. Let* X*,* Y*, and* Z *be vector fields on* M*; let* f *be a smooth function on* M*. Then*

(1) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,

(2) $\nabla_{X+Y}Z = \nabla_X Z + \nabla_Y Z$,

(3) $\nabla_{fX}(Y) = f\nabla_X Y$, *and*

(4) $\nabla_X fY = (Xf)Y + f\nabla_X Y$.

Conversely, if $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(X, Y) \mapsto \nabla_X Y$ *is an operation that satisfies the above four conditions, then there exists a unique linear connection on* M *whose covariant derivative equals* ∇.

Remark 2.44. Proposition 2.43 shows that the space $\mathrm{Conn}(M)$ of linear connections on M is in a bijective correspondence with the set of covariant derivatives in TM.

Definition 2.45. Let Γ be a linear connection in M with torsion form Θ and curvature form Ω. The *torsion* T of Γ is defined by

$$T(X, Y) = u(2\Theta(\xi, \eta))$$

for $X, Y \in T_xM$, where $u \in LM$ and $\pi(u) = x$; ξ and η are vectors of LM at u with $\pi_{*,u}\xi = X$, $\pi_{*,u}\eta = Y$. Similarly, the *curvature* R of Γ is defined as

$$R(X, Y)Z = u(2\Omega(\xi, \eta))(u^{-1}Z).$$

Theorem 2.46. [13, vol. 1, III, Theorem 5.1] *The torsion* T *and curvature* R *can be expressed in terms of the covariant derivatiation on vector fields as*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$
$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$

We consider a tensor of type (r, s) at point $x \in M$ as a multilinear mapping of $T_xM \times \cdots \times T_xM$ (s-times) into $T_xM \times \cdots \times T_xM$ (r-times).

Definition 2.47. Let K be a tensor field of type (r, s) in M. The *covariant differential* ∇K of K is a tensor field of type $(r, s + 1)$ defined as

$$(\nabla K)(X_1, \ldots, X_s; X) = (\nabla_X K)(X_1, \ldots, X_s).$$

Proposition 2.48. [13, vol. 1, III, Proposition 2.10] *If* K *is a tensor field of type* (r, s), *then*

$$(\nabla K)(X_1, \ldots, X_s; X) = \nabla_X(K(X_1, \ldots, X_s)) - \sum_{i=1}^{s} K(X_1, \ldots, \nabla_X X_i, \ldots, X_s).$$

Let M be a smooth manifold equipped with a linear connection ∇. We fix the notation $I = (a, b) \subset \mathbb{R}$, where $-\infty \leq a < b \leq \infty$ till the end of this section.

Definition 2.49. A curve $c : I \to M$ of class C^1 in M is called a *geodesic* of ∇ if

$$\nabla_{\dot{c}(t)} \dot{c}(t) = 0$$

for all $t \in I$. If c is a geodesic, then any reparametrization τ which makes $c(\tau)$ into a geodesic is called an *affine parameter*.

Theorem 2.50. *For any $x \in M$ and for any vector $X \in T_x M$, there is a unique geodesic with the initial condition (x, X); that is, a unique geodesic $c : I \to M$ such that $c(0) = x$ and $\dot{c}(0) = X$.*

Definition 2.51. A linear connection on M is said to be *complete* if every geodesic can be extended to a geodesic $c(\tau)$, defined for $-\infty < \tau < \infty$, where τ is an affine parameter. When the linear connection is complete, we can define the *exponential map* exp at each point $x \in M$ as follows. For each $X \in T_x M$, let $c : I \to M$ be the geodesic with the initial condition (x, X) as in Theorem 2.49. We set

$$\exp X = c(1).$$

Let ∇ and ∇' be two linear connections with respective torsion T and T'. We consider the *difference* $\delta(X, Y) = \nabla_X Y - \nabla'_X Y$ of the two connections, whose symmetrization δ^+ and the anti-symmetrization δ^- are, as usual,

$$\delta^{\pm}(X, Y) = \delta(X, Y) \pm \delta(Y, X).$$

δ is then the sum of its symmetric part and anti-symmetric part, and it vanishes if and only if both its symmetric and anti-symmetric parts vanish. Notice in particular that $T - T' = \delta^-$.

Proposition 2.52. [24, vol. 2, Chapter 6, I, Proposition 14, Corollary 15] *Let δ be the difference of two linear connections. Then the two connections*

(*a*) *are identical if and only if both $\delta^{\pm} = 0$,*

(*b*) *have the same geodesics if and only if $\delta^+ = 0$, and*

(*c*) *have the same torsion if and only if $\delta^- = 0$.*

Consequently, two linear connections have the same geodesics and torsion if and only if they are identical.

Remark 2.53. It follows from Proposition 2.52 that for every linear connection on M, there exists a unique connection with the same geodesics and with zero torsion. We may hence divide $\mathrm{Conn}(M)$ into equivalence classes up to the same geodesics, so that each class has exactly one torsion-free connection. Moreover, if ∇ is a connection with zero torsion and U is a skew-symmetric tensor of type (2, 0), then Proposition 2.52 implies that there is a connection ∇' in the same class as ∇, but with torsion U:

$$\nabla'_X Y = \nabla_X Y + \tfrac{1}{2} U(X, Y).$$

2.3 AFFINE TRANSFORMATIONS

Let (M, Γ) and (M', Γ') be manifolds provided with linear connections Γ and Γ', respectively. Throughout this section, we consider bundles of linear frames LM and LM' of M and M', respectively, so that the structure groups are $G = GL_n\mathbb{R}$ and $G' = GL_{n'}\mathbb{R}$, where $n = \dim M$ and $n' = \dim M'$; likewise, we denote by ∇ and ∇', T and T', R and R' the covariant differentiation, torsion tensor fields, and the curvature tensor fields of (M, Γ) and (M', Γ'), respectively. Let $f : M \to M'$ be a differentiable function of class C^1; it induces a continuous mapping $f_* : TM \to TM'$.

Definition 2.54. We call $f : (M, \Gamma) \to (M', \Gamma')$ an *affine mapping* if $f_* : TM \to TM'$ maps every horizontal curve in (M, Γ) into a horizontal curve in (M', Γ'); that is, if f maps each parallel vector fields along each curve c of (M, Γ) into a parallel vector field along the curve $f(c)$. A diffeomorphism f of M onto itself is called an *affine transformation* of M if it is an affine mapping.

Notation. The set of affine transformations of M with respect to the linear connection Γ forms a group under composition, and it will be denoted by $\mathrm{Aff}(M, \Gamma)$.

Any transformation in M induces in a natural manner an automorphism of the linear frame bundle LM.

Definition 2.55. Let f be a transformation in M, and let $u = (X_1, \cdots, X_n)$ be a linear frame at $x \in M$. The *natural lift* $^\sharp f : LM \to LM$ of f is defined by

$$^\sharp f(u) = (f_{*,x}X_1, \ldots, f_{*,x}X_n).$$

Proposition 2.56. [13, vol. 1, VI, Proposition 1.4, Theorem 1.5] *Let Γ be a linear connection on M.*

(*a*) $\mathrm{Aff}(M,\Gamma)$ *acts on LM simply-transitively, via natural lifts, if M is connected.*

(*b*) *If M has only finite number of connected components, then* $\mathrm{Aff}(M,\Gamma)$ *is a Lie group with respect to the compact-open topology.*

Definition 2.57. A vector field X on M is called an *infinitesimal affine transformation* of Γ if, for every $x \in M$, the local 1-parameter group of local transformations induced by X near x preserves the connection Γ. An infinitesimal affine transformation X of M is said to be *complete* if it generates a global 1-parameter group of affine transformations of M.

Proposition 2.58. [13, vol. 1, VI, Theorem 2.4] *Let Γ be a complete linear connection on M. Then every infinitesimal affine transformation X of M is complete.*

2.3.1 Invariance by parallelism.

Definition 2.59. A linear connection in M is said to be *invariant under parallelism* if, for arbitrary points x and y of M and for an arbitrary curve c from x to y, there exists a (unique) local affine isomorphism f such that $f(x) = y$ and $f_{*,x} = \tau_c$.

The following theorem due to Cartan is an important but difficult result.

Theorem 2.60. [13, vol. 1, VI, Theorem 7.4, Theorem 7.8] *Let M and M' be differentiable manifolds with linear connections. Let $T, R,$ and ∇ (resp. T', R', and ∇') be the torsion, the curvature, and the convariant differentiation of M (resp. M'). Assume $\nabla T = 0$, $\nabla R = 0$ and $\nabla' T' = 0$, $\nabla' R' = 0$.*

(*a*) *If F is a linear ismorphism of T_xM onto T_yM' and maps the tensors T_x and R_x at x into the tensors T'_y and R'_y respectively, then there is an affine isomorphism f of a neighborhood U of x to a neighborhood V of y such that $f(x) = y$ and $f_{*,x} = F$.*

(*b*) *If, in addition, M and M' are connected, simply-connected and complete, then there exists a unique affine transformation f of M to M' such that $f(x) = y$ and $f_{*,x} = F$.*

Corollary 2.61. [13, vol. 1, VI, Corollary 7.9] *Let M be a connected, simply-connected manifold with a complete linear connection such that $\nabla T = 0$ and $\nabla R = 0$. If F is a linear isomorphism of T_xM onto T_yM which maps the tensors T_x and R_x into T_y and R_y, respectively, then there is a unique affine transformation f of M such that $f(x) = y$ and $f_{*,x} = F$. In particular, the group $\mathrm{Aff}(M)$ of affine transformations of M acts transitively on M.*

Theorem 2.62. [13, vol. 1, VI, Corollary 7.5, Theorem 7.7] *Let M be a differentiable manifold with a linear connection such that $\nabla T = 0$ and $\nabla R = 0$.*

(*a*) *For any two points x and y of M, there exists an affine isomorphism of a neighborhood of x onto a neighborhood of y.*

(*b*) *With respect to the atlas consisting of normal coordinate systems, M is an analytic manifold and the connection is analytic.*

Proposition 2.63. [13, vol. 1, VI, Corollary 7.6] *A linear connection is invariant under parallelism if and only if $\nabla T = 0$ and $\nabla R = 0$.*

2.3.2 Symmetric spaces.

Fix a linear connection Γ on M. Let exp denote the exponential map of M with respect to Γ.

Definition 2.64. A *symmetry* s_x at a point $x \in M$ is a diffeomorphism of a neighborhood onto itself which sends $\exp X$ into $\exp(-X)$ for $X \in T_xM$. Since the symmetry at x defined in one neighborhood of x and the symmetry at x defined in another neighborhood of x coincide in their intersection, we can legitimately speak of *the symmetry* at x.

Definition 2.65. If s_x is an affine transformation with respect to Γ for every $x \in M$, then (M, Γ) is said to be an *affine locally symmetric space*. The pair (M, Γ) is said to be an *affine symmetric space* if, for each $x \in M$, the symmetry s_x can be extended to a global affine transformation of M with respect to Γ. If the connection Γ involved is clear from the context, the pair (M, Γ) satisfying the above conditions is then called a *locally symmetric space* or *symmetric space*, respectively.

Theorem 2.66. [13, vol. 2, XI, Theorems 1.1, 1.2, 1.3, 1.4]

(a) *A manifold with a linear connection is locally symmetric if and only if* $T = 0$ *and* $\nabla R = 0$.

(b) *A complete, connected, and simply-connected locally symmetric space is a symmetric space; conversely, every symmetric space is complete.*

(c) *On every connected symmetric space, the group of affine transformations acts transitively.*

Corollary 2.67. *Let* M *be a connected and simply-connected manifold, and let* Γ *be a complete linear connection on* M. *Then* $T = 0$ *and* $\nabla R = 0$ *if and only if* (M, Γ) *is a symmetric space. Furthermore, if the equivalent conditions are satisfied, then the action of* $\mathrm{Aff}(M, \Gamma)$ *on* M *is transitive.*

Proof. If (M, Γ) is a symmetric space, then it is locally symmetric, so that $T = 0$ and $\nabla R = 0$ follows by Theorem 2.66(a). Conversely, if $T = 0$ and $\nabla R = 0$, then M is locally symmetric by Theorem 2.66(a). The conclusion then follows from the completeness of the connection and the simply-connectedness of M, due to Theorem 2.66(b). Furthermore, if the equivalent conditions are satisfied, then Theorem 2.66(c) implies that $\mathrm{Aff}(M, \Gamma)$ acts on M transitively. □

Example 2.68. A Riemannnian manifold is said to be *Riemannian* (*locally*) *symmetric* if it is affine (locally) symmetric with respect to the Riemannian connection. Basic examples of Riemannian symmetric spaces are [13, vol. 2, XI, Section 10] Euclidean space, spheres, projective spaces, and hyperbolic spaces, each with their standard Riemannian metrics.

Let M be a smooth manifold with a linear connection Γ. Putting together Proposition 2.63 and Corollary 2.67, we see that under some topological conditions (M

connected and simply-connected), (M, Γ) with Γ complete and invariant under parallelism is a natural generalization of the concept of symmetric space. The study of the pair (M, Γ) with Γ invariant under parallelism will be our focus in what follows. Furthermore, if M is connected and simply-connected, and if Γ is complete, then $\nabla T = 0$ and $\nabla R = 0$ is a sufficient condition for the transitivity of the $\mathrm{Aff}(M, \Gamma)$-action on M is, according to Corollary 2.61. We hence have, under these assumptions, that $\mathrm{Aff}(M, \Gamma)$ acts on M transitively and Γ is $\mathrm{Aff}(M, \Gamma)$-invariant. This leads us naturally to study invariant linear connections on homogeneous spaces.

2.4 INVARIANT LINEAR CONNECTIONS ON HOMOGENEOUS SPACES

We shall assume throughout this section that K is a connected Lie group acting transitively on the left on a smooth manifold M by

$$\cdot : K \times M \to M, (f, x) \mapsto f \cdot x.$$

We fix base point $o \in M$ and we let H be the stabilizer of o. There exists a diffeomorphism $M \simeq K/H$ and every element $f \in K$ may be viewed as a diffeomorphism

$$L_f : x \mapsto L_f(x) = f \cdot x$$

in M; in particular, $L_f(o) = o$ for all $f \in H$. We shall denote $\mathfrak{k}$ and $\mathfrak{h}$ as the Lie algebras of K and H, respectively.

Definition 2.69. A smooth manifold M with a transitive left K-action is called a *homogeneous K-space*. H described as above is called the *isotropy subgroup* of K with respect to the base point o. Let $u_o \in \pi^{-1}(o) \in P$ be a reference point. We define the *isotropy representation* $\lambda : H \to G$, depending on the choice of u_o, as follows. For each $h \in H$, $h(u_o)$ is a point in the same fibre as u_o, and hence is of the form $h(u_o) = u_o \cdot a$ with some $a \in G$. We define $\lambda(h) = a$.

Remark 2.70. λ is indeed a group homomorphism; see e.g. [13, vol 1, p. 105]. Its differentiability [11, Theorem 2.6] is also a standard fact.

Definition 2.71. A *K-homogeneous principal G-bundle* is a principal G-bundle $\pi : P \to M$, together with left K-actions on P and M, such that the following conditions are satisfied:

(i) the action on M is transitive,

(ii) the projection π is K-equivariant, and

(iii) the actions of G and K on P commute.

Definition 2.72. Let $\pi : P \to M$ be a K-homogeneous G-principal bundle. A connection Γ in P is said to be *K-invariant* if its horizontal subspaces are invariant under the action of K.

Notation. The set of K-invariant connections in P is denoted by $\mathrm{Conn}_K(P, M)$.

Example 2.73. The linear frame bundle LM on a homogeneous space $M \simeq K/H$ is a prototypical example of K-homogeneous principal G-bundle. It is in this case that the K-action on M lifts to LM naturally: for $u \in LM$, $x = \pi(u) \in M$, and $f \in K$,

$$f \cdot u = L_{f_{*,x}}(u). \tag{2.4.1}$$

Moreover, the bundle map $\sharp(u) : K \to LM$, $f \mapsto L^f(u)$ satisfies $\pi \circ \sharp(u) = \pi_K$, where K is viewed as the principal H-bundle $\pi_K : K \to M \simeq K/H$. One also obtains the *isotropy representation* of H, defined as

$$\lambda : H \to GL(T_oM),\ h \mapsto (L_h)_{*,o}.$$

Remark 2.74. It follows from Theorem 2.56(a) that if M is connected, then $\sharp(u)$ gives rise to a *bundle isomorphism* $K \simeq {}^\sharp K(u)$. In this important special case, we shall denote the subbundle by ${}^\sharp K \cdot u$. Moreover, the two K-actions involved give rise to the following induced vector fields ${}^\flat : \mathfrak{k} \to \mathfrak{X}(M)$ and ${}^\sharp : \mathfrak{k} \to \mathfrak{X}(LM)$:

$${}^\flat X_m = \frac{d}{dt}|_{t=0} \exp_K tX \cdot m, \quad {}^\sharp X_u = \frac{d}{dt}|_{t=0} \exp_K tX \cdot u.$$

Definition 2.75. A linear connection Γ on M is said to be *K-invariant* if its horizontal subspaces are invariant under the action of K, in the sense that

$$L^f_{*,u}\Gamma(u) \subset \Gamma(f \cdot u)$$

for all $f \in K$ and $u \in LM$.

Example 2.76. Let K be a connected Lie group and H a closed subgroup of K. If there exists a $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$, then the $\mathfrak{h}$-component of the canonical 1-form of K [13, p. 41] with respect to the decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$ defines a connection in the principal H-bundle $\pi_K : K \to K/H$ which is invariant by the left translations of K. Conversely, any connection in $\pi_K : K \to K/H$ invariant by the left translations of K determines an $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$ and its connection 1-form is obtained in the manner described above.

We state the fundamental theorem of K-invariant connections in a homogeneous principal G-bundle P on the homogeneous space $M \simeq K/H$.

Notation. The set of linear maps $\Lambda : \mathfrak{k} \to \mathfrak{g}$ that satisfies for all $h \in H$

$$\Lambda \circ \mathrm{Ad}(h) = \mathrm{Ad}(\lambda(h)) \circ \Lambda$$

is hereby denoted by $\mathrm{Hom}_H(\mathfrak{k}, \mathfrak{g})$. The subset of those maps that also satisfy $\Lambda|_{\mathfrak{h}} = \lambda_*$ is denoted by $\mathrm{Hom}_H(\mathfrak{k}, \mathfrak{g})_{\lambda_*}$.

Theorem 2.77. [13, vol. 2, X, Theorem 1.2] *Let P be a K-homogeneous principal G-bundle on the homogeneous space $M \simeq K/H$. There is a bijective correspondence*

$$\mathrm{Conn}_K(P; M) \leftrightarrow \mathrm{Hom}_H(\mathfrak{k}, \mathfrak{g})_{\lambda_*}.$$

Definition 2.78. If $\Gamma \in \mathrm{Conn}_K(P; M)$, then the corresponding $\Lambda \in \mathrm{Hom}_H(\mathfrak{k}, \mathfrak{g})_{\lambda_*}$ described in Theorem 2.77 will be referred to as the *connection map* of Γ.

Remark 2.79. Let $X \in \mathfrak{k}$, let $\widetilde{X}$ be the induced vector field on P via the group action of K, and let $u_o \in P$. Then the connection map Λ is given explicitly in terms of the connection form ω [13, vol. 1, Chapter II, Theorem 11.5] by

$$\Lambda : X \mapsto \omega(u_o)\widetilde{X}_{u_o}. \tag{2.4.2}$$

In the special case where $P = LM$ and $G = GL(T_oM) \simeq GL_n\mathbb{R}$, where $o \in M$ is the point corresponding to H, the bijective correspondence in Theorem 2.77 reads, upon fixing $u_o \in \pi^{-1}(o)$, $\mathrm{Conn}_K(M) \leftrightarrow \mathrm{Hom}_H(\mathfrak{k}, \mathfrak{gl}(T_oM))_{\lambda_*}$. This bijection may be described in terms of the natural lift, as follows. If $\omega \in \mathrm{Conn}_K(M)$, then

$$\Lambda : X \mapsto u_o \circ \omega(u_o)^\sharp X_{u_o} \circ u_o^{-1}$$

gives rise to an element $\Lambda \in \mathrm{Hom}_H(\mathfrak{k}, \mathfrak{gl}(T_o M))_{\lambda_*}$ [13, vol. 1, II, Proposition 11.3]. Conversely, if $\Lambda \in \mathrm{Hom}_H(\mathfrak{k}, \mathfrak{gl}(T_o M))_{\lambda_*}$, then the induced map

$$\omega(u_o) : T_{u_o} LM \to GL_n\mathbb{R}, \quad \omega(u_o)^\sharp X \mapsto u_o^{-1} \circ \Lambda X \circ u_o$$

is proved [13, vol. 1, II, Theorem 11.5] to yield $\omega \in \mathrm{Conn}_K(M)$.

Proposition 2.80. *Let $(f, f') : P'(M', G') \to P(M, G)$ be a K-invariant bundle homomorphism between two K-homogeneous principal bundles.*

(*a*) *If $\Gamma' \in \mathrm{Conn}_K(P', M')$, then $f_*\Gamma' \in \mathrm{Conn}_K(P; M)$.*

(*b*) *If Λ' is the connection map of Γ', then $f'_* \circ \Lambda'$ is the connection map of $f_*\Gamma'$.*

Proof. It follows from Proposition 2.37 that $f_*\Gamma' \in \mathrm{Conn}(P; M)$, and we are only left to show the K-invariance of the new connection, i.e., $k \cdot f_{*,u'}\Gamma'(u') = f_{*,k\cdot u'}\Gamma'(k \cdot u')$ for all $k \in K$. But this follows readily from the product rule, and that f is a K-invariant bundle morphism of G-principal bundles, as

$$k \cdot f'_{*,u'} X' = \frac{d}{dt} k \cdot f(\phi^{X'}(t))|_{t=0} = f(\phi^{k\cdot X'}(t))|_{t=0} = f_{*,k\cdot u'} k \cdot X',$$

where $X' \in \Gamma'(u')$, $\phi^{X'}(t)$ is the integral curve to X' starting at u', and $\phi^{k\cdot X'}(t)$ likewise denotes the integral curve of $k \cdot X'$ starting at $k \cdot u'$. This proves part (a); the verification of (b) follows from Theorem 2.37 since

$$f'_* \Lambda' X = f'_*(\omega' \widetilde{X}) = (f'_* \omega')\widetilde{X}$$

for all $X \in \mathfrak{k}$. □

2.4.1 Reductive homogeneous spaces.

Definition 2.81. We say that a homogeneous space $M \simeq K/H$ is *reductive* if

$$\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m},$$

where $\mathfrak{m}$ is an $\mathrm{Ad}|_H$-invariant subspace in the sense that $\mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m}$ for all $h \in H$. The above direct sum decomposition is called an $\mathrm{Ad}|_H$*-invariant decomposition* of $\mathfrak{k}$.

Remark 2.82.

(1) In a reductive homogeneous space $M \simeq K/H$ with $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$, the subspace $\mathfrak{m}$ can be identified with T_oM via $X \mapsto {}^\flat X_o$. Under this identification the isotropy representation, which we shall still denote by λ, is then given by the adjoint action $\mathrm{Ad}|_H : H \to GL(\mathfrak{m})$.

(2) Let $M \simeq K/H$ be a reductive homogeneous space with $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$. We denote $[\ ,\ ]_{\mathfrak{m}} : \mathfrak{k} \bigotimes \mathfrak{k} \to \mathfrak{m}$ and $[\ ,\ ]_{\mathfrak{h}} : \mathfrak{k} \bigotimes \mathfrak{k} \to \mathfrak{h}$ to be the components of the Lie bracket $[\ ,\ ]$ with respect to the decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$.

(3) It is clear that each $\Lambda \in \mathrm{Hom}_H(\mathfrak{k}, \mathfrak{gl}(\mathfrak{m}))_{\lambda_*}$ satisfies

$$\Lambda|_{\mathfrak{m}} \circ \mathrm{Ad}(h) = \mathrm{Ad}(\lambda(h)) \circ \Lambda|_{\mathfrak{m}}$$

for all $h \in H$; conversely, such a restricted map determines Λ uniquely. We shall denote the set of maps $\phi : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ that make the diagram

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\phi"] \arrow[d, "\mathrm{Ad}(h)"'] & \mathfrak{gl}(\mathfrak{m}) \arrow[d, "\mathrm{Ad}(\lambda(h))"] \\ \mathfrak{m} \arrow[r, "\phi"] & \mathfrak{gl}(\mathfrak{m}) \end{tikzcd}$$

commutative for all $h \in H$ by $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$.

Theorem 2.83. [22, vol. 2, X, Theorem 5.1] *Let $M \simeq K/H$ be a reductive homogeneous space with an $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$. There is a bijective correspondence between*

(1) $\mathrm{Conn}_K(M)$,

(2) $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$, *and*

(3) $\mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$.

Remark 2.84. The correspondence between (2) and (3) comes from the $\bigotimes$-Hom adjunction. The H action on $\mathfrak{m} \bigotimes \mathfrak{m}$ is the diagonal adjoint action, and

$$\mathfrak{m} \bigotimes \mathfrak{m} = \Lambda^2\mathfrak{m} \bigoplus S^2\mathfrak{m}$$

as H-modules. Hence (3) may be further decomposed into

$$\mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m}) = \mathrm{Hom}_H(\Lambda^2\mathfrak{m}, \mathfrak{m}) \bigoplus \mathrm{Hom}_H(S^2\mathfrak{m}, \mathfrak{m}).$$

Corollary 2.85. [13, vol. 2, X, Corollary 2.2, Proposition 2.3, Theorem 4.1] *Let $M \simeq K/H$ be a reductive homogeneous space with an $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$. Let $\Lambda \in \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$, and let Z be a vector field in M. Set $A_Z = L_Z - \nabla_Z$, where ∇ is the covariant differentiation determined by the invariant connection corresponding to Λ, and L_Z denotes the Lie differentiation with respect to Z.*

(a) Λ and ∇ are related by $\Lambda X = -(A_X)_o$ for $X \in \mathfrak{m}$.

(b) The torsion tensor T and the curvature tensor R at o of the invariant connection corresponding to Λ can be expressed for all $X, Y \in \mathfrak{m}$ as

$$T(X,Y)_o = (\Lambda X)Y - (\Lambda Y)X - [X,Y]_{\mathfrak{m}},$$

$$R(X,Y)_o = [\Lambda X, \Lambda Y] - \Lambda[X,Y]_{\mathfrak{m}} - \mathrm{ad}[X,Y]_{\mathfrak{h}}.$$

(c) The Lie algebra of the holonomy group $\Phi(u_o)$ of the invariant connection defined by Λ is given by

$$\mathrm{Lie}(\Phi(u_o)) = \mathfrak{m}_0 + [\Lambda\mathfrak{m}, \mathfrak{m}_0] + [\Lambda\mathfrak{m}, [\Lambda\mathfrak{m}, \mathfrak{m}_0]] + \cdots,$$

where $\mathfrak{m}_0$ is the subspace of $\mathfrak{g}$ spanned by $[\Lambda\mathfrak{m}, \Lambda\mathfrak{m}] - \Lambda[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}} - \mathrm{ad}[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}$.

Corollary 2.86. *Let $M \simeq K/H$ be a reductive homogeneous space with an $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$, and let $\Lambda - \Lambda' \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$. Then the K-invariant connections correspond to Λ, Λ' have*

(a) the same geodesics if and only if $\Lambda - \Lambda' \in \mathrm{Hom}_H(\Lambda^2\mathfrak{m}, \mathfrak{m})$, or

(b) the same torsion if and only if $\Lambda - \Lambda' \in \mathrm{Hom}_H(S^2\mathfrak{m}, \mathfrak{m})$.

Proof. This follows readily from Proposition 2.52 and Corollary 2.85(a). □

We will later need a version of Corollary 2.85 (a) in a slightly more general context. Let $M \simeq K/H$ be a reductive homogeneous space with an $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$. Let P be a K-homogeneous principal G-bundle, $\Gamma \in \mathrm{Conn}_K(P; M)$, and $\Lambda \in \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{g})$ the corresponding connection map of Γ. Let $\rho : G \to GL(V)$ be a finite-dimensional representation of G and let $E = P \times_\rho V$ be the associated vector bundle, and ∇ the corresponding covariant derivative acting on sections of E.

Theorem 2.87. *If $A_Z = L_Z - \nabla_Z$, where L_Z is the Lie differentiation with respect to Z, and $o \in M$ is the point with isotropy group H, then Λ and ∇ are related by*

$$(\rho_* \circ \Lambda)X = -(A_X)_o.$$

Proof. The proof of [13, vol. II, Corollary 1.3], which is stated and proved for $P = LM$ and $\rho = \mathrm{id}$, carries over to this more general context. $\square$

2.4.2 Canonical connections.

Let $M \simeq K/H$ be a reductive homogeneous space with respect to the $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$. We recall that every K-invariant connection in LM corresponds to an element of $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$. In particular, there is a connection that corresponds to $0 \in \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$.

Definition 2.88. The K-invariant connection on $M \simeq K/H$ that corresponds to the zero map in $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ is called the *canonical connection* relative to the $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$, and will be denoted by $\Gamma_{\mathfrak{m}}$.

Notation. $\mathrm{Can}_K(M)$ will denote the set of K-invariant canonical connections in M.

Remark 2.89. $\mathrm{Can}_K(M)$ is in bijection, via the correspondence $\Gamma_{\mathfrak{m}} \leftrightarrow \mathfrak{m}$, with the set of $\mathrm{Ad}|_H$-invariant complements of $\mathfrak{h} \subset \mathfrak{k}$.

Proposition 2.90. [13, vol. 2, X, Proposition 2.4, Corollary 2.5, Proposition 2.7]

(*a*) *The canonical connection $\Gamma_{\mathfrak{m}}$ is the unique K-invariant connection possessing the following property. Let $\exp_K tX$ be the 1-parameter subgroup of K generated by an arbitrary element $X \in \mathfrak{m}$. Then the orbit ${}^{\sharp}\exp_K tX(u_o)$ is horizontal.*

(*b*) *For each $X \in \mathfrak{m}$, the curve $\exp_K tX \cdot o$ is a geodesic and, conversely, every geodesic through o is of the form $\exp_K tX \cdot o$ for some $X \in \mathfrak{m}$. Moreover, along the geodesics the parallel transport is given by*

$$\tau_{\exp_K tX \cdot o}|_s^0 = (\exp_K sX)_*.$$

(*c*) *If a tensor field on M is K-invariant, then it is parallel with respect to the canonical connection $\Gamma_{\mathfrak{m}}$.*

Proposition 2.91. [13, vol. 2, X, Corollary 2.5, Corollary 4.3]

(a) *The canonical connection* $\Gamma_{\mathfrak{m}}$ *is complete and invariant under parallelism.*

(b) *The torsion tensor* T *and the curvature tensor* R *of* $\Gamma_{\mathfrak{m}}$ *satisfy*

$$T(X,Y)_o = -[X,Y]_{\mathfrak{m}} \text{ and } R(X,Y)_o = -\mathrm{ad}[X,Y]_{\mathfrak{h}}.$$

(c) *The Lie algebra of the holonomy group* $\Phi(u_o)$ *of the canonical connection is*

$$\mathrm{Lie}\Phi(u_o) = \mathrm{span}_{\mathbb{R}}\mathrm{ad}[\mathfrak{m},\mathfrak{m}]_{\mathfrak{h}}.$$

The subject of research in this work is the pair (M,Γ), where M is a connected and simply-connected manifold, and Γ is a complete linear connection that is invariant under parallelism. Under these conditions, the following fundamental theorem shows that M is a reductive homogeneous space and Γ is an invariant canonical connection.

Theorem 2.92. [13, vol. 2, X, Theorem 2.8] *Let* M *be a connected manifold with a linear connection* Γ*. Let* $LM(u_o)$ *be the holonomy bundle through* $u_o \in LM$*.*

(a) *If* K *is a connected subgroup of* $\mathrm{Aff}(M,\Gamma)$ *such that* ${}^\sharp K \cdot u_o \supseteq LM(u_o)$*, then* M *is a reductive homogeneous* K*-space and* $\Gamma \in \mathrm{Can}_K(M)$*.*

(b) *If* M *is simply-connected,* Γ *is complete, and* $\nabla T = 0$*,* $\nabla R = 0$*, then* M *is a reductive homogeneous* $\mathrm{Aff}(M,\Gamma)^0$*-space and* $\Gamma \in \mathrm{Can}_{\mathrm{Aff}(M,\Gamma)^0}(M)$*. Moreover, the smallest* $K \leq \mathrm{Aff}(M,\Gamma)^0$ *such that* M *is a reductive homogeneous* K*-space and* $\Gamma \in \mathrm{Can}_K(M)$ *satisfies* ${}^\sharp K \cdot u_o = LM(u_o)$*.*

Proposition 2.93. [13, vol. 2, X, Theorem 2.9] *Let* $M \simeq K/H$ *be a reductive homogeneous* K*-space relative to the* $\mathrm{Ad}|_H$*-invariant decomposition* $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$*. The unique torsion free* K*-invariant connection with the same geodesics as* $\Gamma_{\mathfrak{m}}$ *is the connection corresponding to the map* $\frac{1}{2}[\cdot,\cdot]_{\mathfrak{m}} \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$*.*

Definition 2.94. The K-invariant connection corresponding to the map $\frac{1}{2}[\cdot,\cdot]_{\mathfrak{m}} \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$ is called the *naturally torsion-free connection* associated to $\mathfrak{m}$.

3.0 LINES OF K-INVARIANT CONNECTIONS INVARIANT UNDER PARALLELISM

Throughout this chapter $M \simeq K/H$ will denote a reductive homogeneous space. We also fix a base point $o \in M$ and $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$, an $\mathrm{Ad}|_H$-invariant decomposition. This assures, in particular, that $\mathrm{Can}_K(M) \neq \varnothing$ as $\Gamma_{\mathfrak{m}}$ is a canonical connection.

A fundamental question asks for the description of the set $\mathrm{Can}_K(M)$. In linear algebraic terms the set $\mathrm{Can}_K(M)$ is in bijection with the set of $\mathrm{Ad}|_H$-invariant complements of $\mathfrak{h}$ inside $\mathfrak{k}$. It seems that this point of view does not offer much insight into the structure of the set $\mathrm{Can}_K(M)$.

A related set of connections is $\mathrm{Conn}_K^{=}(M)$, the set of K-invariant connections on M that are invariant under parallelism. From Proposition 2.90 we know that

$$\mathrm{Can}_K(M) \subset \mathrm{Conn}_K^{=}(M).$$

There are topological obstructions for the equality of the two sets; for example, canonical connections are always complete. Nevertheless, if M is connected, simply connected, and $\Gamma \in \mathrm{Conn}_K^{=}(M)$ is complete, then we know by Theorem 2.92 that

$$\Gamma \in \mathrm{Can}_{\mathrm{Aff}(M,\Gamma)^0}(M) \text{ and } K \leq \mathrm{Aff}(M,\Gamma)^0.$$

The connection Γ determines, by Theorem 2.92, an $\mathrm{Iso}(M,\Gamma)$-invariant decomposition

$$\mathfrak{aff}(M,\Gamma) = \mathfrak{iso}(M,\Gamma) \bigoplus \mathfrak{c},$$

where $\mathfrak{aff}(M,\Gamma)$ and $\mathfrak{iso}(M,\Gamma)$ are the Lie algebras of $\mathrm{Aff}(M,\Gamma)^0$ and $\mathrm{Iso}(M,\Gamma)$, its subgroup fixing o. Then $\Gamma \in \mathrm{Can}_K(M)$ if and only if $\mathfrak{c} \subset \mathfrak{k}$. In explicit examples, this condition can be examined with the help of the Ambrose-Singer theorem (Corollary 2.85). Another possible line of inquiry is the following. Remark that

$$\mathrm{Aff}(M,\Gamma)^0 = K \cdot \mathrm{Iso}(M,\Gamma).$$

Oniščik [20] classified all triples (G, G', G'') of reductive groups such that $G = G'G''$. The classification can be potentially used to describe the set of complete K-invariant connections, invariant under parallelism with reductive affine transformation and isotropy groups on a connected, simply-connected K-homogeneous space.

The set $\mathrm{Conn}_K^{=}(M)$ is, on the other hand, amenable to be studied with algebraic techniques. From Wang's theorem Theorem 2.83 we know that $\mathrm{Conn}_K(M)$ is parametrized by the space $\mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$. Let us denote by $\mathrm{Hom}_H^{=}(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$ the subset of $\mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$ corresponding to $\mathrm{Conn}_K^{=}(M)$. The condition that a connection is invariant under parallelism translates into an algebraic condition for the corresponding linear map. This makes it possible for us to address some structural questions regarding the set $\mathrm{Hom}_H^{=}(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$, such as under which conditions does it contain lines.

3.1 COVARIANT CROSS-DERIVATION OF TORSION AND CURVATURE TENSOR FIELDS

Definition 3.1. Let $\eta \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$. If $\eta \in \mathrm{Hom}_H(S^2\mathfrak{m}, \mathfrak{m})$, then η is said to be *symmetric*; if $\eta \in \mathrm{Hom}_H(\Lambda^2\mathfrak{m}, \mathfrak{m})$, then η is said to be *skew-symmetric*. The *symmetrization* η^+ and the *skew-symmetrization* η^- of η are defined to be

$$\eta^{\pm}(X,Y) = \eta(X,Y) \pm \eta(Y,X).$$

In the multilinear situation, $\eta^{\pm}$ will refer to

$$\eta^{\pm}(X,Y,Z_1,\cdots,Z_s) = \eta(X,Y,Z_1,\cdots,Z_s) \pm \eta(Y,X,Z_1,\cdots,Z_s).$$

Notation.

(1) In view of the bijective correspondence between spaces $\mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$ and $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ in Theorem 2.83, we associate for each $\alpha \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$ the map $\alpha \in \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ where

$$\alpha(X)Y = \alpha(X,Y). \tag{3.1.1}$$

Throughout, α is, depending on the context, regarded as an element of either set.

(2) We denote the covariant derivative corresponding to $\eta \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$ by ∇^η. The one corresponding to $\eta = 0$ is, in particular, denoted by ∇^0.

Definition 3.2. Let $\beta : \bigotimes_{i=1}^s \mathfrak{m} \to \mathfrak{m}$ be a linear map. A map $\delta \in \mathrm{Hom}(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ is said to be a *derivation of* $(\mathfrak{m}, \beta)$ if

$$\delta(Z)\beta(X_1, \ldots, X_s) = \beta(\delta(Z)X_1, \cdots, X_s) + \cdots + \beta(X_1, \cdots, \delta(Z)X_s).$$

Notation. We denote by $\mathrm{Der}(\beta)$ the set of derivations of $(\mathfrak{m}, \beta)$. If $\beta_1, \cdots, \beta_n$ are linear maps as above, then we denote

$$\mathrm{Der}(\beta_1, \cdots, \beta_n) = \mathrm{Der}(\beta_1) \bigcap \cdots \bigcap \mathrm{Der}(\beta_n).$$

Definition 3.3. Let $\alpha : \mathfrak{m} \bigotimes \mathfrak{m} \to \mathfrak{m}$ and $\beta : \bigotimes_{i=1}^s \mathfrak{m} \to \mathfrak{m}$ be linear maps. We define the *formal differentiation* of β with respect to α by

$$d_\alpha \beta(X_1, \cdots, X_s; Z) = \alpha(Z)\beta(X_1, \cdots, X_s) - \sum_{i=1}^s \beta(X_1, \cdots, \alpha(Z)X_i, \cdots, X_s).$$

Remark 3.4. This definition is motivated by Proposition 2.48. Let α, β be as in Definition 3.3. Note that $d_\alpha \beta = 0$ if and only if $\alpha \in \mathrm{Der}(\beta)$.

Definition 3.5. We assign to $\eta \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$ the 3-*cyclic symmetrization*

$$\mathrm{Cyc}(\eta)(X, Y, Z) = \eta(X, \eta(Y, Z)) + \eta(Y, \eta(Z, X)) + \eta(Z, \eta(X, Y)).$$

Definition 3.6. We define, for all $X, Y \in \mathfrak{m}$,

$$\gamma_{\mathfrak{m}}(X, Y) = [X, Y]_{\mathfrak{m}} \text{ and } \gamma_{\mathfrak{h}}(X, Y) = [X, Y]_{\mathfrak{h}}$$

according to the convention in Remark 2.82.

Remark 3.7. $\gamma_{\mathfrak{m}} \in \mathrm{Hom}_H(\Lambda^2 \mathfrak{m}, \mathfrak{m})$ and so it corresponds to a K-invariant connection $\nabla^{\gamma_{\mathfrak{m}}}$ that will be called the *bracket connection associated to* $\mathfrak{m}$.

Proposition 3.8. *Let* $\eta \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$. *Then for all* $X, Y \in \mathfrak{m}$,

$$\nabla^\eta_X Y = \nabla^0_X Y + \eta(X)Y.$$

Proof. Corollary 2.85(a) shows that, under the identification $\mathfrak{m} \simeq T_o M$,

$$\eta(X)Y = \nabla^\eta_X Y - L_X Y.$$

This in turn gives $\nabla^0_X Y = L_X Y$ by substituting in $\eta = 0$. $\square$

Remark 3.9. Let $\eta \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$. The torsion tensor T^η and the curvature tensor R^η of ∇^η can be expressed, by Corollary 2.85(b), for all $X, Y, Z \in \mathfrak{m}$ as

$$\begin{aligned} T^\eta(X,Y) &= \eta(X,Y) - \eta(Y,X) - [X,Y]_\mathfrak{m}, \\ R^\eta(X,Y)Z &= \eta(X,\eta(Y,Z)) - \eta(Y,\eta(X,Z)) - \eta([X,Y]_\mathfrak{m}, Z) - [[X,Y]_\mathfrak{h}, Z]. \end{aligned}$$

This shows that for $\eta = 0$, the torsion and curvature tensors are

$$T^0(X,Y) = -[X,Y]_\mathfrak{m},\ R^0(X,Y,Z) = -[[X,Y]_\mathfrak{h}, Z].$$

The above definition hence gives

$$T^\eta = \eta^- + T^0, \quad R^\eta = (\eta \circ (\mathrm{id} \otimes \eta))^- - \eta \circ (\gamma_\mathfrak{m} \otimes \mathrm{id}) + R^0. \tag{3.1.2}$$

Proposition 3.10. *Let $\alpha, \beta \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$.*

(a) *If α is a canonical connection, then $\nabla^\alpha T^\beta = 0$ and $\nabla^\alpha R^\beta = 0$.*

(b) *$\nabla^\alpha T^\beta = d_\alpha T^\beta$ and $\nabla^\alpha R^\beta = d_\alpha R^\beta$ hold identically.*

Proof. Part (a) is a direct consequence of Proposition 2.90(c) and Proposition 2.91(a). To prove part (b), recall that

$$\begin{aligned} \nabla^\alpha T^\beta(X,Y;Z) =& \nabla^\alpha_Z T^\beta(X,Y) - T^\beta(\nabla^\alpha_Z X, Y) - T^\beta(X, \nabla^\alpha_Z Y), \\ \nabla^\alpha R^\beta(X,Y,Z;W) =& \nabla^\alpha_W R^\beta(X,Y,Z) \\ & - R^\beta(\nabla^\alpha_W X, Y, Z) - R^\beta(X, \nabla^\alpha_W Y, Z) - R^\beta(X,Y,\nabla^\alpha_W Z), \end{aligned}$$

where each individual term is given by, for the torsion parts

$$\begin{aligned} \nabla^\alpha_W T^\beta(X,Y) =& \nabla^0_W T^\beta(X,Y) + \alpha(W) T^\beta(X,Y), \\ T^\beta(\nabla^\alpha_W X, Y) =& T^\beta(\nabla^0_W X, Y) + T^\beta(\alpha(W)X, Y), \\ T^\beta(X, \nabla^\alpha_W Y) =& T^\beta(X, \nabla^0_W Y) + T^\beta(X, \alpha(W)Y), \end{aligned}$$

and likewise for the curvature parts,

$$
\begin{aligned}
\nabla^{\alpha}_W R^{\beta}(X,Y,Z) =& \nabla^0_W R^{\beta}(X,Y,Z) + \alpha(W)R^{\beta}(X,Y,Z);\\
R^{\beta}(\nabla^{\alpha}_W X,Y,Z) =& R^{\beta}(\nabla^0_W X,Y,Z) + R^{\beta}(\alpha(W)X,Y,Z),\\
R^{\beta}(X,\nabla^{\alpha}_W Y,Z) =& R^{\beta}(X,\nabla^0_W Y,Z) + R^{\beta}(X,\alpha(W)Y,Z),\\
R^{\beta}(X,Y,\nabla^{\alpha}_W Z) =& R^{\beta}(X,Y,\nabla^0_W Z) + R^{\beta}(X,Y,\alpha(W)Z).
\end{aligned}
$$

This shows, by summarizing the above identities, that

$$
\begin{aligned}
\nabla^{\beta} T^{\beta}(X,Y;Z) =& \nabla^0 T^{\eta}(X,Y;Z) + d_{\alpha}T^{\beta}(X,Y;Z),\\
\nabla^{\beta} R^{\beta}(X,Y,Z;W) =& \nabla^0 R^{\beta}(X,Y,Z;W) + d_{\alpha}R^{\beta}(X,Y,Z;W).
\end{aligned}
$$

The conclusions now follow, as $\nabla^0 T^{\beta} = 0$ and $\nabla^0 R^{\beta} = 0$ hold by part (a). $\square$

Corollary 3.11. *Let* $\eta \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$. *Then*

$$
\begin{aligned}
\nabla^0_Z T^{\eta}(X,Y) &= T^{\eta}(\nabla^0_Z X,Y) + T^{\eta}(X,\nabla^0_Z Y),\\
\nabla^0_W R^{\eta}(X,Y,Z) &= R^{\eta}(\nabla^0_W X,Y,Z) + R^{\eta}(X,\nabla^0_W Y,Z) + R^{\eta}(X,Y,\nabla^0_W Z).
\end{aligned}
$$

Proof. Proposition 2.48 shows that for $X,Y,Z \in \mathfrak{m}$,

$$
\begin{aligned}
\nabla^0 T^{\eta}(X,Y;Z) &= \nabla^0_Z T^{\eta}(X,Y) - T^{\eta}(\nabla^0_Z X,Y) - T^{\eta}(X,\nabla^0_Z Y),\\
\nabla^0 R^{\eta}(X,Y,Z;W) &= \nabla^0_W R^{\eta}(X,Y,Z)\\
&\quad - R^{\eta}(\nabla^0_W X,Y,Z) - R^{\eta}(X,\nabla^0_W Y,Z) - R^{\eta}(X,Y,\nabla^0_W Z).
\end{aligned}
$$

The conclusion follows from $\nabla^0 T^{\eta} = 0$ and $\nabla^0 R^{\eta} = 0$. $\square$

Definition 3.12. We assign to each pair $\alpha, \beta \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$ the following *associated linear maps*:

$$
\begin{aligned}
A_{\alpha} &= d_{\alpha}T^0, B_{\alpha\beta} = d_{\alpha}\beta^-,\\
C_{\alpha} =& d_{\alpha}R^0, D_{\alpha\beta} = -d_{\alpha}(\beta \circ (\gamma_{\mathfrak{m}} \otimes \mathrm{id})), E_{\alpha\beta} = d_{\alpha}(\beta \circ (\mathrm{id} \otimes \beta))^-.
\end{aligned}
$$

Proposition 3.13. *Let $\alpha, \beta \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$. Then for all $X, Y, Z, W \in \mathfrak{m}$,*

$$
\begin{aligned}
A_\alpha(X,Y,Z) &= -\alpha(Z)[X,Y]_{\mathfrak{m}} + [\alpha(Z)X,Y]_{\mathfrak{m}} + [X,\alpha(Z)Y]_{\mathfrak{m}},\\
B_{\alpha\beta}(X,Y,Z) &= \alpha(Z,\beta(X,Y)) - \alpha(Z,\beta(Y,X)) - \beta(\alpha(Z,X),Y)\\
&\quad + \beta(Y,\alpha(Z,X)) - \beta(X,\alpha(Z,Y)) + \beta(\alpha(Z,Y),X),\\
C_\alpha(X,Y,Z,W) &= -\alpha(W)[[X,Y]_{\mathfrak{h}},Z]\\
&\quad + [[\alpha(W)X,Y]_{\mathfrak{h}},Z] + [[X,\alpha(W)Y]_{\mathfrak{h}},Z] + [[X,Y]_{\mathfrak{h}},\alpha(W)Z],\\
D_{\alpha\beta}(X,Y,Z,W) &= -\alpha(W)\beta([X,Y]_{\mathfrak{m}},Z) + \beta([\alpha(W)X,Y]_{\mathfrak{m}},Z)\\
&\quad + \beta([X,\alpha(W)Y]_{\mathfrak{m}},Z) + \beta([X,Y]_{\mathfrak{m}},\alpha(W)Z),\\
E_{\alpha\beta}(X,Y,Z,W) &= \alpha(W)\beta(X,\beta(Y,Z)) - \beta(\alpha(W)X,\beta(Y,Z))\\
&\quad - \beta(X,\beta(\alpha(W)Y,Z)) - \beta(X,\beta(Y,\alpha(W)Z))\\
&\quad - \alpha(W)\beta(Y,\beta(X,Z)) + \beta(\alpha(W)Y,\beta(X,Z))\\
&\quad + \beta(Y,\beta(\alpha(W)X,Z)) + \beta(Y,\beta(X,\beta(W)Z)).
\end{aligned}
$$

Proof. It follows directly from Definition 3.3 that

$$
\begin{aligned}
A_\alpha(X,Y,Z) &= d_\alpha T^0(X,Y;Z)\\
&= -\alpha(Z)[X,Y]_{\mathfrak{m}} + [\alpha(Z)X,Y]_{\mathfrak{m}} + [X,\alpha(Z)Y]_{\mathfrak{m}},\\
B_{\alpha\beta}(X,Y,Z) &= d_\alpha\beta(X,Y;Z) - d_\alpha\beta(Y,X;Z)\\
&= -\beta(X,\alpha(Z,Y)) - \alpha(Z,\beta(Y,X)) + \beta(Y,\alpha(Z,X))\\
&\quad + \beta(\alpha(Z,Y),X) + \alpha(Z,\beta(X,Y)) - \beta(\alpha(Z,X),Y),\\
C_\alpha(X,Y,Z,W) &= d_\alpha R^0(X,Y,Z;W)\\
&= -\alpha(W)[[X,Y]_{\mathfrak{h}},Z]\\
&\quad + [[\alpha(W)X,Y]_{\mathfrak{h}},Z] + [[X,\alpha(W)Y]_{\mathfrak{h}},Z] + [[X,Y]_{\mathfrak{h}},\alpha(W)Z],\\
D_{\alpha\beta}(X,Y,Z,W) &= -d_\alpha(\beta\circ(\gamma_{\mathfrak{m}}\otimes \mathrm{id}))(X,Y,Z;W)\\
&= -\alpha(W)\beta([X,Y]_{\mathfrak{m}},Z) + \beta([\alpha(W)X,Y]_{\mathfrak{m}},Z)\\
&\quad + \beta([X,\alpha(W)Y]_{\mathfrak{m}},Z) + \beta([X,Y]_{\mathfrak{m}},\alpha(W)Z),
\end{aligned}
$$

$$
\begin{aligned}
E_{\alpha\beta}(X,Y,Z,W) = & d_\alpha(\beta\circ(\mathrm{id}\otimes\beta))(X,Y,Z;W) - d_\alpha(\beta\circ(\mathrm{id}\otimes\beta))(Y,X,Z;W) \\
= & -\beta(\alpha(W)X,\beta(Y,Z)) - \beta(X,\beta(\alpha(W)Y,Z)) \\
& + \alpha(W)\beta(X,\beta(Y,Z)) - \beta(X,\beta(Y,\alpha(W)Z)) \\
& - \alpha(W)\beta(Y,\beta(X,Z)) + \beta(Y,\beta(\alpha(W)X,Z)) \\
& + \beta(\alpha(W)Y,\beta(X,Z)) + \beta(Y,\beta(X,\alpha(W)Z)).
\end{aligned}
$$

□

Remark 3.14. Note that $C_\alpha = d_\alpha(\mathrm{ad}\circ(\gamma_{\mathfrak{h}}\otimes\mathrm{id}))$.

Proposition 3.15. *If* $\alpha,\beta\in\mathrm{Hom}_H(\mathfrak{m}\bigotimes\mathfrak{m},\mathfrak{m})$, *then*

$$
\nabla^\alpha T^\beta = A_\alpha + B_{\alpha\beta}, \quad \nabla^\alpha R^\beta = C_\alpha + D_{\alpha\beta} + E_{\alpha\beta}.
$$

Proof. Since $T^\beta = \beta^- + T^0$ and $R^\beta = (\beta\circ(\mathrm{id}\otimes\beta))^- - \beta\circ(\gamma_{\mathfrak{m}}\otimes\mathrm{id}) + R^0$,

$$
\begin{aligned}
\nabla^\alpha T^\beta = & d_\alpha T^\beta = d_\alpha\beta^- + d_\alpha T^0 = B_{\alpha\beta} + A_\alpha, \\
\nabla^\alpha R^\beta = & d_\alpha R^\beta = d_\alpha(\beta\circ(\mathrm{id}\otimes\beta))^- + d_\alpha(-\beta\circ(\gamma_{\mathfrak{m}}\otimes\mathrm{id})) + d_\alpha R^0 = D_{\alpha\beta} + E_{\alpha\beta} + C_\alpha,
\end{aligned}
$$

follow from Remark 3.9 and Proposition 3.10. □

Corollary 3.16. *Let* $\alpha,\beta\in\mathrm{Hom}_H(\mathfrak{m}\bigotimes\mathfrak{m},\mathfrak{m})$. *Then bilinear maps* $A_\alpha(\cdot,\cdot,Z)$, $B_{\alpha\beta}(\cdot,\cdot,Z)$, $C_\alpha(\cdot,\cdot,Z,W)$, $D_{\alpha\beta}(\cdot,\cdot,Z,W)$, *and* $E_{\alpha\beta}(\cdot,\cdot,Z,W)$ *are skew-symmetric.*

Proof. The statements regarding bilinear maps $A_\alpha(\cdot,\cdot,Z)$, $B_{\alpha\beta}(\cdot,\cdot,Z)$, $C_\alpha(\cdot,\cdot,Z,W)$, and $D_{\alpha\beta}(\cdot,\cdot,Z,W)$ are clear from the above expressions. To prove the skew-symmetry of $E_{\alpha\beta}(\cdot,\cdot,Z,W)$, observe that

$$
\begin{aligned}
E_{\alpha\beta}(X,Y,Z,W) = & \beta(\alpha(W)Y,\beta(X,Z)) - \beta(\alpha(W)X,\beta(Y,Z)) \\
& -\beta(X,\beta(\alpha(W)Y,Z)) + \beta(Y,\beta(\alpha(W)X,Z)) \\
& +\alpha(W)\beta(X,\beta(Y,Z)) - \alpha(W)\beta(Y,\beta(X,Z)) \\
& -\beta(X,\beta(Y,\alpha(W)Z)) + \beta(Y,\beta(X,\alpha(W)Z)),
\end{aligned}
$$

which is a sum of four skew-symmetric (with respect to X and Y) linear maps. □

Definition 3.17. Let $S\subset \mathrm{Hom}_H(\mathfrak{m}\bigotimes\mathfrak{m},\mathfrak{m})$. Then

$$c(S)=|\mathrm{Hom}_H^{\equiv}(\mathfrak{m}\bigotimes\mathfrak{m},\mathfrak{m})\bigcap S|$$

counts the number of connections invariant under parallelism parametrized by S.

Proposition 3.18. *If $\eta\in\mathrm{Hom}_H(\mathfrak{m}\bigotimes\mathfrak{m},\mathfrak{m})$, then*

$$\nabla^{t\eta}T^{t\eta}=tA_\eta+t^2B_{\eta\eta},\quad \nabla^{t\eta}R^{t\eta}=tC_\eta+t^2D_{\eta\eta}+t^3E_{\eta\eta}.$$

Proof. We substitute $\alpha=\beta=t\eta$ into Proposition 3.15, so that $\nabla^{t\eta}T^{t\eta}=A_{t\eta}+B_{t\eta t\eta},\nabla^{t\eta}R^{t\eta}=C_{t\eta}+D_{t\eta t\eta}+E_{t\eta t\eta}$, where $A_{t\eta}=tA_\eta, B_{t\eta t\eta}=t^2B_{\eta\eta}, C_{t\eta}=tC_\eta, D_{t\eta t\eta}=t^2D_{\eta\eta}$, and $E_{t\eta t\eta}=t^3E_{\eta\eta}$ follow by easy computation. □

Corollary 3.19. *Let $0\neq\eta\in\mathrm{Hom}_H(\mathfrak{m}\bigotimes\mathfrak{m},\mathfrak{m})$. The followings are equivalent:*

(a) $\mathbb{R}\eta\subset\mathrm{Hom}_H^{\equiv}(\mathfrak{m}\bigotimes\mathfrak{m},\mathfrak{m})$.

(b) $c(\mathbb{R}\eta)\geq 4$.

(c) *The maps A_η, $B_{\eta\eta}$, C_η, $D_{\eta\eta}$, and $E_{\eta\eta}$ are identically zero.*

Proof. Simply examine the number of zeros of the two polynomials in Theorem 3.18. □

Theorem 3.20. *Let $\eta\in\mathrm{Hom}_H(\mathfrak{m}\bigotimes\mathfrak{m},\mathfrak{m})$.*

(a) $\eta\in\mathrm{Hom}_H(\mathfrak{m},\mathrm{Der}(\gamma_{\mathfrak{m}}))$ *if and only if* $A_\eta=0$.

(b) *If* $\eta\in\mathrm{Hom}_H(\Lambda^2\mathfrak{m},\mathfrak{m})$, *then* $B_{\eta\eta}=2\mathrm{Cyc}(\eta)$; *or* $B_{\eta\eta}=0$ *if* $\eta\in\mathrm{Hom}_H(S^2\mathfrak{m},\mathfrak{m})$.

(c) $\eta\in\mathrm{Hom}_H(\mathfrak{m},\mathrm{Der}(\mathrm{ad}\circ(\gamma_{\mathfrak{h}}\otimes\mathrm{id})))$ *if and only if* $C_\eta=0$.

(d) *If* $\eta\in\mathrm{Hom}_H(\mathfrak{m},\mathrm{Der}(\eta,\gamma_{\mathfrak{m}}))$, *then* $D_{\eta\eta}=0$.

(e) *If* $\eta\in\mathrm{Hom}_H(\mathfrak{m},\mathrm{Der}(\eta))$, *then* $E_{\eta\eta}=0$.

Proof. Parts (a) and (c) hold since $A_\eta=-d_\eta\gamma_{\mathfrak{m}}$ and $C_\eta=d_\eta(\mathrm{ad}\circ(\gamma_{\mathfrak{h}}\otimes\mathrm{id}))$. The statement ($b$) is clear for, if η is skew-symmetric, then it equals half of its skew-symmetrization; that is,

$$B_\eta=d_\eta\eta^-=2d_\eta\eta=2\mathrm{Cyc}(\eta).$$

On the other hand, if η is symmetric, then it is skew-symmetrizes to zero, and $B_{\eta\eta} = d_\eta \eta^- = 0$. For (d), if $\eta \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\eta))$, then

$$\eta(W)\eta([X,Y]_{\mathfrak{m}}, Z) = \eta(\eta(W)[X,Y]_{\mathfrak{m}}, Y) + \eta([X,Y]_{\mathfrak{m}}, \eta(W)Z).$$

Since $\eta \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\gamma_{\mathfrak{m}}))$, the first term on the right side splits into

$$\eta([\eta(W)X, Y]_{\mathfrak{m}}, Y) + \eta([X, \eta(W)Y]_{\mathfrak{m}}, Y),$$

which shows that $D_{\eta\eta} = 0$ identically. The statement (e) is also clear as, by Proposition 3.15, we may put

$$\begin{aligned}
E^1_{\eta\eta}(X,Y,Z,W) &= \eta(W)\eta(X, \eta(Y,Z)) \\
&\quad - \eta(\eta(W)X, \eta(Y,Z)) - \eta(X, \eta(\eta(W)Y, Z)) - \eta(X, \eta(Y, \eta(W)Z)), \\
E^2_{\eta\eta}(X,Y,Z,W) &= -\eta(W)\eta(Y, \eta(X,Z)) \\
&\quad + \eta(\eta(W)Y, \eta(X,Z)) + \eta(Y, \eta(\eta(W)X, Z)) + \eta(Y, \eta(X, \eta(W)Z)),
\end{aligned}$$

showing that $E_{\eta\eta} = E^1_{\eta\eta} + E^2_{\eta\eta}$ does vanish if η is a derivation of $(\mathfrak{m}, \eta)$. □

Corollary 3.21. *If $\eta \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\eta, \gamma_{\mathfrak{m}}, \mathrm{ad} \circ (\gamma_{\mathfrak{h}} \otimes \mathrm{id})))$, then*

$$\mathbb{R}\eta \subset \mathrm{Hom}_H^{=}(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m}).$$

Proof. Theorem 3.20 shows that $\nabla^{t\eta} T^{t\eta} = 0$ and $\nabla^{t\eta} R^{t\eta} = 0$ for all t. □

3.3 LINES IN $\mathrm{HOM}_H^{=}(\Lambda^2\mathfrak{M}, \mathfrak{M})$.

Proposition 3.22. *Let $\eta \in \mathrm{Hom}_H(\Lambda^2\mathfrak{m}, \mathfrak{m})$.*

(a) *If $\alpha \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$ represents a canonical connection, then $\nabla^\alpha \eta = 0$.*

(b) *$\eta \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\eta))$ if and only if $B_{\eta\eta} = 0$.*

Proof. Theorem 3.20(b) implies, by Remark 3.9, that $T^\eta = 2\eta + T^0$, hence

$$0 = \nabla^\alpha T^\eta = \nabla^\alpha(2\eta + T^0) = 2\nabla^\alpha \eta.$$

This concludes (a), and part (b) holds for, by skew-symmetry of η, $\mathrm{Cyc}(\eta) = 0$ precisely when η is derivation of $(\mathfrak{m}, \eta)$. $\square$

Theorem 3.23. *Let* $0 \neq \eta \in \mathrm{Hom}_H(\Lambda^2\mathfrak{m}, \mathfrak{m})$ *such that*

$$\eta \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\eta, \mathrm{ad} \circ (\gamma_{\mathfrak{h}} \otimes \mathrm{id}))).$$

Then $\mathbb{R}\eta \subset \mathrm{Hom}_H^{=}(\Lambda^2\mathfrak{m}, \mathfrak{m})$ *if and only if* $c(\mathbb{R}\eta) \geq 2$.

Proof. Suppose $0 \neq \eta \in \mathrm{Hom}_H(\Lambda^2\mathfrak{m}, \mathfrak{m})$. If $\eta \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\eta, \mathrm{ad} \circ (\gamma_{\mathfrak{h}} \otimes \mathrm{id})))$, then Theorem 3.20 and Proposition 3.22 show that $C_\eta = 0$, $E_{\eta\eta} = 0$, and $B_{\eta\eta} = 0$. Hence $c(\mathbb{R}\eta) \geq 2$ implies $\nabla^{t\eta} T^{t\eta} = tA_\eta = 0$ and $\nabla^{t\eta} R^{t\eta} = t^2 D_{\eta\eta} = 0$ for some $t \neq 0$, so that both A_η and $D_{\eta\eta}$ must vanish. This shows that $\mathbb{R}\eta \subset \mathrm{Hom}_H^{=}(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$. $\square$

Theorem 3.24. *Let* $\eta \in \mathrm{Hom}_H(\Lambda^2\mathfrak{m}, \mathfrak{m})$ *such that*

$$\eta \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\gamma_{\mathfrak{m}})).$$

Then $\mathbb{R}\eta \subset \mathrm{Hom}_H(\Lambda^2\mathfrak{m}, \mathfrak{m})$ *if and only if* $c(\mathbb{R}\eta) \geq 2$.

Proof. Suppose $\eta \in \mathrm{Hom}_H(\Lambda^2\mathfrak{m}, \mathfrak{m})$. If $\eta \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\gamma_{\mathfrak{m}}))$, then $A_\eta = 0$ by Theorem 3.20. Under this condition, $c(\mathbb{R}\eta) \geq 2$ gives some $t \neq 0$ such that

$$\nabla^{t\eta} T^{t\eta} = t^2 B_{\eta\eta} = 0 \text{ and } \nabla^{t\eta} R^{t\eta} = t(C_\eta + tD_{\eta\eta} + t^2 E_{\eta\eta}) = 0.$$

But this forces $B_{\eta\eta} = 0$ which, by Proposition 3.22, implies $\eta \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\eta))$. Hence $D_{\eta\eta}$ and $E_{\eta\eta}$ both vanish by Theorem 3.20, so that $C_\eta = 0$, concluding that $\mathbb{R}\eta \subset \mathrm{Hom}_H^{=}(\Lambda^2\mathfrak{m}, \mathfrak{m})$. $\square$

3.4 LINES IN HOM$_H^{=}(S^2\mathfrak{M}, \mathfrak{M})$.

Theorem 3.25. *Let* $0 \neq \eta \in \mathrm{Hom}_H(S^2\mathfrak{m}, \mathfrak{m})$ *such that* $\eta \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\eta, \gamma_{\mathfrak{m}}))$. *Then the followings are equivalent:*

(a) $\mathbb{R}\eta \subset \mathrm{Hom}_H^{=}(S^2\mathfrak{m}, \mathfrak{m})$.

(b) $c(\mathbb{R}\eta) \geq 2$.

(c) $\eta \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\mathrm{ad} \circ (\gamma_{\mathfrak{h}} \otimes \mathrm{id})))$.

Proof. For the implication from (b) to (a), observe that since η is symmetric, $B_{\eta\eta} = 0$ by Lemma 3.22. If $\eta \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\gamma_{\mathfrak{h}}, \gamma_{\mathfrak{m}}))$, then $A_\eta = 0$, $D_{\eta\eta} = 0$, and $E_{\eta\eta} = 0$ hold by Theorem 3.20. This shows that, indeed, $\nabla^{t\eta} T^{t\eta} = 0$ and $\nabla^{t\eta} R^{t\eta} = tC_\eta$ for all t. As $c(\mathbb{R}\eta) \geq 2$, this implies $C_\eta = 0$. The implication from (a) to (b) is obvious. The equivalence of (a) and (c) follows from Corollary 3.21. □

3.5 THE BRACKET CONNECTION ASSOCIATED TO $\mathfrak{M}$.

As we can see from the results in this chapter, the existence of lines of connections invariant under parallelism is tied to the existence of certain algebraic structures on $\mathfrak{m}$. In full generality, the only line in $\mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$ that can be considered is the line spanned by $\gamma_{\mathfrak{m}} \in \mathrm{Hom}_H(\Lambda^2\mathfrak{m}, \mathfrak{m})$ (when $\gamma_{\mathfrak{m}} \neq 0$).

Remark 3.26. Symmetries exist on $\mathbb{R}\gamma_{\mathfrak{m}}$ with respect to the point $\frac{1}{2}\gamma_{\mathfrak{m}}$: the torsion tensors satisfy $T^{(t+\frac{1}{2})\gamma_{\mathfrak{m}}} = -T^{(\frac{1}{2}-t)\gamma_{\mathfrak{m}}}$ and both $(\frac{1}{2} \pm t)\gamma_{\mathfrak{m}}$ have the same Ricci tensor.

Proposition 3.27. *Let $M \simeq K/H$ be a reductive homogeneous K-space with the $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$. Then $\gamma_{\mathfrak{m}} \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\gamma_{\mathfrak{m}}))$ if and only if every element in $\mathbb{R}\gamma_{\mathfrak{m}}$ has an auto-parallel torsion field.*

Proof. Since $A_{\gamma_{\mathfrak{m}}} = d_{\gamma_{\mathfrak{m}}} T^0 = -d_{\gamma_{\mathfrak{m}}} \gamma_{\mathfrak{m}} = -\mathrm{Cyc}(\gamma_{\mathfrak{m}})$, by skew-symmetry of $\gamma_{\mathfrak{m}}$, Proposition 3.22 yields $B_{\gamma_{\mathfrak{m}}} = 2\mathrm{Cyc}(\gamma_{\mathfrak{m}}) = -2A_{\gamma_{\mathfrak{m}}}$. Proposition 3.15 then gives

$$\nabla^{\gamma_{\mathfrak{m}}} T^{\gamma_{\mathfrak{m}}} = A_{\gamma_{\mathfrak{m}}} + B_{\gamma_{\mathfrak{m}}} = -A_{\gamma_{\mathfrak{m}}}. \tag{3.5.1}$$

The conclusion now follows from Proposition 3.20. □

Theorem 3.28. *Let $M \simeq K/H$ be a reductive homogeneous K-space with the $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$. The following statements are equivalent:*

(a) $\gamma_{\mathfrak{m}} \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\gamma_{\mathfrak{m}}, \mathrm{ad} \circ (\gamma_{\mathfrak{m}} \otimes \mathrm{id})))$.

(b) $\gamma_{\mathfrak{m}}$ is invariant under parallelism.

(c) $\mathbb{R}\gamma_{\mathfrak{m}} \subset \mathrm{Hom}_H^{=}(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$.

Proof. Suppose (a) holds. Then Proposition 3.20 and (4.1.1) show that linear maps $A_{\gamma_\mathfrak{m}}, B_{\gamma_\mathfrak{m}\gamma_\mathfrak{m}}, C_{\gamma_\mathfrak{m}}, D_{\gamma_\mathfrak{m}\gamma_\mathfrak{m}}$, and $E_{\gamma_\mathfrak{m}\gamma_\mathfrak{m}}$ are identically zero, proving (b). Conversely, if (b) holds, then we utilize a simple fact that $R^{\gamma_\mathfrak{m}} = \mathrm{Cyc}_\mathfrak{m} + R^0$ to write

$$\nabla^{\gamma_\mathfrak{m}} R^{\gamma_\mathfrak{m}} = d_{\gamma_\mathfrak{m}} \mathrm{Cyc}(\gamma_\mathfrak{m}) + C_{\gamma_\mathfrak{m}}, \tag{3.5.2}$$

which concludes, in accompany with (4.1.1) and Theorem 3.20, that $A_{\gamma_\mathfrak{m}} = 0$, viz (a) holds. Next, suppose (b) holds. Then by (4.1.1), $\gamma_\mathfrak{m} \in \mathrm{Hom}_H(\mathfrak{m}, \mathrm{Der}(\gamma_\mathfrak{m}))$ must follow. Theorem 3.24 then assures that (c) holds, so that (b) and (c) are indeed equivalent. □

4.0 CARTAN-SCHOUTEN CONNECTIONS

Let S be a Lie group and $K = S \times S$ the direct product of S with itself. We have a natural action

$$K \times S \to S, \quad ((a, b), x) \mapsto axb^{-1}. \tag{4.0.1}$$

The stabilizer in K of the identity $e \in S$ is given by the diagonal group

$$H = \Delta(S \times S) = \{(s, s) | s \in S\}.$$

Hence there exists a homogeneous space structure $S \simeq K/H$, in which the action of $\Delta(S \times S)$ on $T_eS \simeq \mathfrak{s}$ is given by the isotropy representation (Definition 2.69) and

$$\mathfrak{k} = \mathfrak{s} \bigoplus \mathfrak{s}, \quad \mathfrak{h} = \Delta(\mathfrak{s} \bigoplus \mathfrak{s})$$

are the corresponding Lie algebras. This is in fact a *reductive* homogeneous space structure: there exist $\mathrm{Ad}|_{\Delta(S\times S)}$-invariant decompositions, such as

$$\begin{aligned}
\mathfrak{k} &= \mathfrak{h} \bigoplus \mathfrak{m}_+, \text{where } \mathfrak{m}_+ = \{0\} \bigoplus \mathfrak{s}, \\
\mathfrak{k} &= \mathfrak{h} \bigoplus \mathfrak{m}_-, \text{where } \mathfrak{m}_- = \mathfrak{s} \bigoplus \{0\}, \text{or} \\
\mathfrak{k} &= \mathfrak{h} \bigoplus \mathfrak{m}_0, \text{where } \mathfrak{m}_0 = \{(X, -X) : X \in \mathfrak{s}\}.
\end{aligned}$$

The canonical connections Γ_+, Γ_-, and Γ_0 corresponding to these $\mathrm{Ad}_{\Delta(S\times S)}$-invariant decompositions were considered by Cartan and Schouten [6], who called them (+)-*connection*, (−)-*connection*, and (0)-*connection*, respectively.

Definition 4.1. An $(S \times S)$-invariant linear connection on S is called a *bi-invariant* connection. The set of all bi-invariant connections on S is denoted by $\mathrm{Conn}_{S\times S}(S)$.

Henceforth we fix the $\mathrm{Ad}|_{\Delta(S\times S)}$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}_-$.

Proposition 4.2. *The space* $\mathrm{Conn}_{S\times S}(S)$ *may be parametrized by* $\mathrm{Hom}_S(\mathfrak{s}\bigotimes\mathfrak{s},\mathfrak{s})$ *or, equivalently, by* $\mathrm{Hom}_S(\mathfrak{s},\mathfrak{gl}(\mathfrak{s}))$.

Proof. This follows from Theorem 2.83 and that $\Delta(S\times S)\simeq S$ and $\mathfrak{m}_-\simeq\mathfrak{s}$. □

Remark 4.3. Proposition 4.2 shows, in particular, that the space of bi-invariant connections on S is never empty, for the Lie bracket $\mathrm{ad}_{\mathfrak{s}}:(X,Y)\mapsto[X,Y]$ is in the space $\mathrm{Hom}_S(\mathfrak{s}\bigotimes\mathfrak{s},\mathfrak{s})$.

Notation. If $\eta\in\mathrm{Hom}_S(\mathfrak{s}\bigotimes\mathfrak{s},\mathfrak{s})$ (or, equivalently, $\mathrm{Hom}_S(\mathfrak{s},\mathfrak{gl}(\mathfrak{s}))$, then we denote by Γ_η the corresponding element of $\mathrm{Conn}_S(S\times S)$.

Definition 4.4. Let S be a Lie group. Consider the family of $\mathrm{Ad}|_{\Delta(S\times S)}$-invariant complements of $\Delta(\mathfrak{s}\bigoplus\mathfrak{s})$ inside $\mathfrak{s}\bigoplus\mathfrak{s}$:

$$\mathfrak{p}_\alpha=\{(r_\alpha X,s_\alpha X):X\in\mathfrak{s}\},$$

where $r_\alpha=\frac{1}{2}(\alpha+1)$, $s_\alpha=\frac{1}{2}(\alpha-1)$, and $\alpha\in\mathbb{R}$. The corresponding canonical connection $\Gamma_{\mathfrak{p}_\alpha}$ will be called the *Cartan-Shouten connection* of parameter α.

Remark 4.5. $\mathfrak{m}_+$, $\mathfrak{m}_-$, and $\mathfrak{m}_0$ equal $\mathfrak{p}_\alpha$ for $\alpha=-1,1$, and 0, respectively.

Theorem 4.6. *Let* S *be a Lie group. Then*

$$\Gamma_{\mathfrak{p}_\alpha}=\Gamma_{\frac{1}{2}(1-\alpha)\mathrm{ad}_{\mathfrak{s}}}.$$

Proof. By Theorem 2.77 there is a bijective correspondence between the set of bi-invariant connections on S and $\mathrm{Hom}_{\Delta(S\times S)}(\mathfrak{s}\bigoplus\mathfrak{s},\mathfrak{gl}(\mathfrak{s}))_{\lambda_*}$, where λ_* is the tangent map of the $\Delta(S\times S)$-action on $T_eS=\mathfrak{s}$. In our case

$$\lambda_*=\mathrm{ad}_{\mathfrak{s}}.$$

The canonical connection $\Gamma_{\mathfrak{p}_\alpha}$ will correspond to a map Λ_α for which $\Lambda_\alpha|_{\mathfrak{p}_\alpha}\equiv 0$. Using $(X,0)\in\mathfrak{m}_-$, where $X\in\mathfrak{s}$, we write direct sum

$$(X,0)=\tfrac{1}{2}(1-\alpha)(X,X)\oplus(r_\alpha X,s_\alpha X)$$

with respect to the decomposition $\mathfrak{s}\bigoplus\mathfrak{s}=\Delta(\mathfrak{s}\bigoplus\mathfrak{s})\bigoplus\mathfrak{p}_\alpha$. We see thus

$$\Lambda_\alpha(X,0)=\tfrac{1}{2}(1-\alpha)\mathrm{ad}_{\mathfrak{h}}(X,X)=\tfrac{1}{2}(1-\alpha)\mathrm{ad}_{\mathfrak{s}}X.$$

It then follows that, with respect to the parametrization of the space $\mathrm{Conn}_{S\times S}(S)$ by $\mathrm{Hom}_{\Delta(S\times S)}(\mathfrak{m}_-, \mathfrak{gl}(\mathfrak{m}_-))$, Λ_α corresponds to $\frac{1}{2}(1-\alpha)\mathrm{ad}_{\mathfrak{s}}$. $\square$

Corollary 4.7. *Let S be a Lie group.*

(a) The Cartan-Schouten connections on S are precisely the scalings of the bracket connection. In particular, the naturally torsion-free connection $\frac{1}{2}\mathrm{ad}_{\mathfrak{s}}$ is the Cartan-Schouten connection Γ_0.

(b) The torsion tensor and curvature tensor of $\Gamma_{\mathfrak{p}_\alpha}$, respectively, are given by

$$T^\alpha = -\alpha \mathrm{ad}_{\mathfrak{s}}, \quad R^\alpha = \frac{1}{4}(1-\alpha)^2(\mathrm{ad}_{\mathfrak{s}} \circ (\mathrm{id} \otimes \mathrm{ad}_{\mathfrak{s}}))^- - \frac{1}{2}(1-\alpha)\mathrm{ad}_{\mathfrak{s}} \circ (\mathrm{ad}_{\mathfrak{s}} \otimes 1).$$

4.1 BI-INVARIANT CANONICAL CONNECTIONS ON COMPACT LIE GROUPS

It turns out that if S is a simple compact Lie group, then the Cartan-Schouten connections are the only bi-invariant canonical connections on S. We give the proof of this fact in this section.

Lemma 4.8. *Let $\alpha_i \in \mathrm{Hom}_S(\mathfrak{s} \bigotimes \mathfrak{s}, \mathfrak{s})$, $c_i \in \mathbb{R}$ for $1 \le i \le n$, and $c = c_1 + \cdots + c_n$. If $\eta = c_1\alpha_1 + \cdots + c_n\alpha_n$, then*

$$\nabla^\eta T^\eta = \sum_{i,j} c_i c_j \nabla^{\alpha_i} T^{\alpha_j} + (1-c)\nabla^\eta T^0.$$

Proof. By linearity one writes $\nabla^\eta T^\eta = \sum_i c_i \nabla^{\alpha_i} T^\eta$. It is then left to compute $\nabla^{\alpha_i} T^\eta$ for each $1 \le i \le n$. But $T^\eta = \sum_i c_i T^{\alpha_i} + (1-c)T^0$, and the result follows. $\square$

Theorem 4.9. [16, Theorem 6.1] *If S is a compact simple Lie group, then the space of bi-invariant connections on S is one-dimensional, with the exception of SU_n, $n \ge 3$. More precisely,* $\mathrm{Hom}_S(\mathfrak{s} \bigotimes \mathfrak{s}, \mathfrak{s})$ *is spanned by*

$$\gamma_1(X,Y) = [X,Y],$$
$$\gamma_2(X,Y) = i(XY + YX - \frac{2}{n}\mathrm{tr}(XY)I_n)$$

if $S = SU_n$ for $n \ge 3$, or by γ_1 otherwise.

The proof of this theorem is based on the following facts. Recall from Remark 2.84 and Proposition 4.2 that the set $\mathrm{Conn}_{S\times S}(S)$ is parametrized by

$$\mathrm{Hom}_S(\mathfrak{s}\bigotimes\mathfrak{s},\mathfrak{s})=\mathrm{Hom}_S(\Lambda^2\mathfrak{s},\mathfrak{s})\bigoplus\mathrm{Hom}_S(S^2\mathfrak{s},\mathfrak{s}).$$

The dimension of $\mathrm{Hom}_S(\Lambda^2\mathfrak{s},\mathfrak{s})$ is the multiplicity of the adjoint representation of S in $\Lambda^2\mathfrak{s}$, which turns out to be equal to one for any simple compact Lie group S. The dimension of $\mathrm{Hom}_S(S^2\mathfrak{s},\mathfrak{s})$ is the multiplicity of the adjoint representation of S in $S^2\mathfrak{s}$, which, for S simple compact Lie group, is always zero with the exception of SU_n, $n\geq 3$, for which the multiplicity is one. The verification that $\gamma_1\in\mathrm{Hom}_S(\Lambda^2\mathfrak{s},\mathfrak{s})$ and that $\gamma_2\in\mathrm{Hom}_S(\Lambda^2\mathfrak{s},\mathfrak{s})$ for $S=SU_n$, $n\geq 3$ is straightforward.

Theorem 4.10. *Let S be a compact simple Lie group. The only bi-invariant connections invariant under parallelism on S are the Cartan-Schouten connections.*

Proof. Being canonical connections, the Cartan-Schouten connections are invariant under parallelism. We will check that there are no other bi-invariant connections invariant under parallelism. In view of Theorem 4.9, this task is reduced to the examination of the cases $S=SU_n$ for $n\geq 3$; namely, it is required to show that if $\gamma=c_1\gamma_1+c_2\gamma_2$ is invariant under parallelism on S, then $c_2=0$.

Let E_{ij} $(1\leq i,j\leq n)$ denote the $n\times n$ matrix with all entries zero with the exception of the (i,j)-entry which equals 1. Suppose $c_2\neq 0$. If $c_1=0$, then for each $\eta=c_2\gamma_2$ we let $X=E_{12}-E_{21}$, $Y=i(E_{12}+E_{21})$, and $Z=i(E_{11}-E_{22})$. Since γ_2 is symmetric, Theorem 3.18 and part (b) of Proposition 3.22 give $\nabla^\eta T^\eta=c_2A_{\gamma_2}$, where

$$A_{\gamma_2}=-\gamma_2(Z)[X,Y]+[\gamma_2(Z)X,Y]+[X,\gamma_2(Z)Y]=-2i(E_{11}+E_{22}-\frac{2}{n}I_n).\quad(4.1.1)$$

If $c_1\neq 0$, then by Lemma 4.8,

$$\nabla^\eta T^\eta=c_2^2d_{\gamma_2}T^{\gamma_2}+c_1c_2(d_{\gamma_1}T^{\gamma_2}+d_{\gamma_2}T^{\gamma_1})+(1-c_1-c_2)d_\eta T^0.$$

It is clear that $T^{\gamma_2}=-T^{\gamma_1}=T^0$, so that $d_{\gamma_1}T^{\gamma_2}=0$ and $d_\eta T^0=c_2d_{\gamma_2}T^0$ follow by Remark 3.4; hence

$$\nabla^\eta T^\eta=c_2^2d_{\gamma_2}T^0+c_1c_2(d_{\gamma_2}(-T^0))+(1-c_1-c_2)c_2d_{\gamma_2}T^0=c_2(1-2c_1)A_{\gamma_2},$$

which shows, as does the case in (4.1.1), that the the same choices of X, Y, and Z gives the desired conclusion for all $c_1 \neq \frac{1}{2}$.

We are hence left to check that none of the connections of the form $\eta = \frac{1}{2}\mathrm{ad}_{\mathfrak{s}} + t\gamma_2$ is invariant under parallelism for any $t \neq 0$. For that purpose, we consider $X = E_{12} - E_{21}$, $Y = i(E_{12} + E_{21})$, $Z = i(E_{11} + E_{22})$, and $W = -E_{12} + E_{21}$; moreover, we set $G_t^{\pm} = i(\frac{1}{2}X \pm tY)$, $K_t^{\pm} = i(\frac{1}{2}Y \pm tX)$, and $F_q = -2it(E_{11} + E_{22} - \frac{q}{n}I_n)$ for $q \in \mathbb{Z}$. To prove that $\nabla^\eta R^\eta(X, Y, Z; W) \neq 0$, we establish the matrix

$$\begin{bmatrix} \eta & X & Y & Z & W & E_{11} & E_{22} \\ X & -F_2 & Z & Y & F_2 & K_t^- & -K_t^- \\ Y & -Z & -F_4 & X & W & -G_t^- & -G_t^- \\ Z & Y & -X & -F_2 & -Y & -2t(E_{11} - \frac{1}{n}I_n) & 2t(E_{22} - \frac{1}{n}I_n) \\ W & F_2 & -Z & Y & -F_2 & -K_t^+ & i(\frac{1}{2}Y + tW) \\ E_{11} & -K_t^+ & G_t^+ & -2t(E_{11} - \frac{1}{n}I_n) & K_t^- & 2it(E_{11} - \frac{1}{n}I_n) & 0 \\ E_{22} & K_t^+ & -G_t^- & 2t(E_{22} - \frac{1}{n}I_n) & -K_t^+ & 0 & 2it(E_{22} - \frac{1}{n}I_n) \end{bmatrix}.$$

Direct computation using these results shows $\eta(W)R^\eta(X, Y, Z) = -12it^2n^{-1}I_n$, $R^\eta(\eta(W)X, Y, Z) = 0$, $R^\eta(X, \eta(W)Y, Z) = -X$, and $R^\eta(X, Y, \eta(W)Z) = -F_2 + X$. This shows (by substituting $\alpha = \beta = \eta$ in Proposition 3.10)

$$\nabla^\eta R^\eta(X, Y, Z, W) = F_{-4} \neq 0$$

for all $t \neq 0$, as desired. □

Remark 4.11. One could also give a short proof as follows. Since $\eta = c_1\gamma_1 + c_2\gamma_2$ is invariant under parallelism, then by Proposition 3.10(a),

$$0 = \nabla^\eta T^0 = d_\eta T^0 = c_1 d_{\gamma_1} T^0 + c_2 d_{\gamma_2} T^0.$$

This shows, as γ_1 is invariant under parallelism, that $c_2 d_{\gamma_2} T^0 = c_2 A_{\gamma_2} = 0$. We hence conclude from (4.1.1) that indeed $c_2 = 0$.

Theorem 4.12. *Let S be a compact simple Lie group. The only canonical bi-invariant connections on S are the Cartan-Schouten connections.*

Proof. This follows from Theorem 4.10 and that $\mathrm{Can}_{S\times S}(S) \subset \mathrm{Conn}^{=}_{S\times S}(S)$. □

4.2 HORIZONTAL LIFT EQUATION

Let S be a connected Lie group, $\pi : LS \to S$ its linear frame bundle, and $\Gamma_{\mathfrak{p}_\alpha}$ a Cartan-Schouten connection. Let $c : [0,1] \to S$ be an arbitrary C^1 curve and fix $u_0 \in \pi^{-1}(c(0))$. Theorem 2.22 asserts the local existence of a unique horizontal lift c^* of c through u_0 with respect to $\Gamma_{\mathfrak{p}_\alpha}$. In this section we shall be concerned with the derivation of its governing ODE, the horizontal lift equation. We consider the case of a general principal bundle before specializing to Lie groups and Cartan-Schouten connections.

4.2.1 Principal bundles.

Let $\pi : P \to M$ be principal G-bundle equipped with a connection Γ. As horizontal curves are allowed to be piecewise, we assume the existence of a neighborhood $U \subset M$ of $c(0)$ on which P is trivial and which contains the image of the entire curve c. The notation of Section 2.1 and Section 2.2 is in effect.

Proposition 4.13. [24, vol. II, Proposition 8.9] *Let s be a section of P over some open set U that trivializes the bundle. If $g : U \to G$ is smooth, then for all $m \in M$*

$$(R \circ (s,g))_{*,m} = (R_{g(m)})_{*,s(m)} \circ s_{*,m} + {}^{\sigma}(l(g(m)^{-1})_{*,g(m)} \circ g_{*,m})_{s(m)\cdot g(m)}.$$

Proof. Applying chain rule to the composition map $R \circ (s,g) : M \to P$,

$$\begin{aligned}(R \circ (s,g))_{*,m} &= R_{*,(s(m),g(m))} \circ (s_{*,m}, g_{*,m}) \\ &= (R_{g(m)})_{*,s(m)} \circ s_{*,m} + (\sigma_{s(m)})_{*,g(m)} \circ g_{*,m}.\end{aligned}$$

Let $X \in T_m M$ and $Y = l(g(m)^{-1})_{*,g(m)} \circ g_{*,m} X$. Then

$$\begin{aligned}(\sigma_{s(m)})_{*,g(m)} \circ g_{*,m}(X) &= \frac{d}{dt}|_{t=0} s(m) \cdot (g(m) \cdot \exp_G tY) \\ &= \frac{d}{dt}|_{t=0} (s(m) \cdot g(m)) \cdot \exp_G tY \\ &= (\sigma_{s(m)\cdot g(m)})_{*,e} Y := {}^{\sigma}(Y)_{s(m)\cdot g(m)}.\end{aligned}$$

So, $(R \circ (s,g))_{*,m} = (R_{g(m)})_{*,g(m)} \circ s_{*,m} + {}^{\sigma}(l(g(m)^{-1})_{*,g(m)} \circ g_{*,m})$. $\square$

Proposition 4.14. *Let $c : [0,1] \to M$ be a C^1 curve, $s : U \to \pi^{-1}U$ a smooth section, and $\gamma = s \circ c$, and $a = g \circ c$ for some smooth $g : U \to G$. If $u = \gamma \cdot a$, then*

$$u_{*,t} = (R_{a(t)})_{*,\gamma(t)} \circ \gamma_{*,t} + {}^{\sigma}(l(a(t)^{-1})_{*,a(t)} \circ a_{*,t})_{u(t)}.$$

Proof. It follows from Proposition 4.13 by letting $s = \gamma$, $x = c(t)$, and $X = \dot{c}(t)$. □

Definition 4.15. Let P be a principal G-bundle equipped with a connection form ω and $c : [0,1] \to M$ a C^1-curve. If $\gamma = s \circ c$ for some section s in P, then

$$a_{*,t} = -r(a(t))_{*,e}\omega(\gamma(t))\gamma_{*,t}$$

is called the *horizontal lift equation of* ω (*with respect to* γ).

The following result is implicitly used in the proof of [13, vol. I, Proposition 3.1]:

Theorem 4.16. *Let $c : [0,1] \to M$ be a piecewise C^1 curve, and choose $u_0 \in P$ with $\pi(u_0) = c(0)$. Let $\gamma : [0,1] \to P$ be a lift of c. If $a = g \circ c$ for some smooth $g : U \to G$ and satisfies $a_{*,t} = -r(a(t))_{*,e}\omega(\gamma(t))\gamma_{*,t}$, then $u = \gamma \cdot a$ is the unique horizontal lift of c through u_0 with respect to ω.*

Proof. It is enough to show that, by Theorem 2.22, $u = \gamma \cdot a$ is a horizontal curve through u_0 with respect to ω. Indeed, $u(0) = \gamma(0) \cdot a(0) = u_0 \cdot e = u_0$. Moreover, keeping Definition 2.17 in mind, we obtain

$$\begin{aligned} \omega(u(t))u_{*,t} &= \omega(u(t))(R_{a(t)})_{*,\gamma(t)}\gamma_{*,t} + {}^{\sigma}(l(a(t)^{-1})_{*,a(t)} \cdot a_{*,t})_{u(t)}) \\ &= \mathrm{Ad}(a(t)^{-1})\omega(\gamma(t))\gamma_{*,t} + l(a(t)^{-1})a_{*,t} \\ &= l(a(t)^{-1})_{*,a(t)}(r(a(t))_{*,e}\omega(\gamma(t))\gamma_{*,t} + a_{*,t}). \end{aligned}$$

Proposition 4.14 then shows $\dot{u}$ is horizontal with respect to ω everywhere. □

From this point on, we shall assume the setting of the principal $GL_n(\mathbb{R})$-bundle of linear frames over a reductive homogeneous space $M \simeq K/H$ with an $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$ ($n = \dim M$). The equation in Definition 4.15 is then an identity on the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$, which can be written as

$$\dot{a}a^{-1} = -\omega(\gamma)\dot{\gamma}. \tag{4.2.1}$$

Notice that, from the technical point of view, the horizontal lift equation (4.2.1) should be $\dot{a}a^{-1} = \omega(\gamma)\dot{\gamma}^{\perp}$, where $\dot{\gamma}^{\perp}$ is the vertical component of $\dot{\gamma}$ relative to the connection ω. There is, however, no ambiguity in Proposition 4.18: if $f \in K$ and X is a vector field in LM, then it can easily seen that for all $u \in P$,

$$L^f_{*,u}(X_u)^{\perp} = (L^f_{*,u}X)^{\perp}_{f\cdot u}.$$

Definition 4.17. Let $c : [0,1] \to M$ be a C^1 curve. A *K-lift* c_K is a lift of c with respect to the bundle $\pi_K : K \to M \simeq K/H$. (Its existence is justified by the connection described in Example 2.76 and by Theorem 2.22.) The curve $L^{c_K}(u_0)$ is called a *naturally-induced lift* of c.

Proposition 4.18. *Let $\omega \in \mathrm{Conn}_K(M)$, let $c : [0,1] \to M$ be a C^1 curve, choose $u_0 \in P$ with $\pi(u_0) = c(0)$, and set $c(0) \in U$ to be the neighborhood that trivializes LM. If there exists a C^1-section $s : U \to \pi^{-1}U$ such that $L^{c_K}(u_0) = s \circ c$, then the horizontal lift equation of ω with reference curve $L^{c_K}(u_0)$ is equivalent to*

$$\dot{a}a^{-1} = -\omega(u_0)L^{c_K^{-1}}\dot{\gamma}.$$

Proof. Set $\gamma = L^{c_K}(u_0)$ and apply Remark 4.2.1 and K-invariance of ω. $\square$

4.2.2 Cartan-Schouten connections for SU_2.

We identify $\mathbb{R}^4$ with the set of complex 2×2 matrices

$$\mathbb{H} = \{\begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} \mid \alpha, \beta \in \mathbb{C}\}. \tag{4.2.2}$$

In this circumstance, SU_2 may be viewed as the unit 3-sphere in $\mathbb{H}$ with respect to the quadratic form given by determinant. We choose $\nu = \{U_1, U_2, U_3\}$, where $U_1 = i(E_{11} - E_{22})$, $U_2 = -E_{12} + E_{21}$, and $U_3 = i(E_{12} + E_{21})$, to be the standard basis of $\mathfrak{su}(2)$. Then we fix the frame (e, ν) on S. The global trivialization of linear frame bundles over Lie groups (Example 2.5) gives $LSU_2 \simeq SU_2 \times GL_3(\mathbb{R})$. For simplicity, we shall work with the (trivial) bundle

$$\pi : SU_2 \times GL_3(\mathbb{R}) \to SU_2$$

instead of the linear frame bundle. Note that $(e,\nu) \in LSU_2$ corresponds to $(e, I_3) \in SU_2 \times GL_3(\mathbb{R})$.

We then consider the imbeddings $SU_2 \subset \mathbb{R}^4$ and $GL_3(\mathbb{R}) \subset (\mathbb{R}^4)^3$; more precisely, we identify each $a = \sum_{i,j} a^{ij} E_{ij} \in GL_3(\mathbb{R})$ with the 4×3 matrix, with top row being zero and bottom 3×3 minor identical to a, i.e., the matrix ${}^{\dagger}a = \sum_{ij} a^{ij} E_{i+1,j}$. We may hence view $SU_2 \times GL_3(\mathbb{R})$ as a submanifold of the space $\mathbb{R}^4 \bigoplus (\mathbb{R}^4)^3$, or $\mathfrak{gl}(4,\mathbb{R})$. The tangent space $T_{(e,\nu)}(SU_2 \times GL_3(\mathbb{R}))$ may also be viewed as a linear subspace of $\mathfrak{gl}(4,\mathbb{R})$(the different copies of $\mathbb{R}^4$ being seen as columns of a 4×4 real matrix). Secondly, let $K^\alpha(t) = \exp_{SU_2 \times SU_2} t(r_\alpha X, s_\alpha X)$, $X \in \mathfrak{su}(2)$, be the integral curve of $(r_\alpha X, s_\alpha X)$ in $SU_2 \times SU_2$. Each component function of $K^\alpha(t)$ is given by

$$K_1^\alpha = (\epsilon_{r_\alpha t}, \epsilon_{s_\alpha t}),\ K_2^\alpha = p(\epsilon_{r_\alpha t}, \epsilon_{s_\alpha t})p^{-1}, K_3^\alpha = q(\epsilon_{r_\alpha t}, \epsilon_{s_\alpha t})q^{-1},$$

where $\epsilon_t = \mathrm{diag}(e^{it}, e^{-it})$, $U_2 = pU_1p^{-1}$, and $U_3 = qU_1q^{-1}$, with

$$p = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}, q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

We now investigate the decomposition

$$T_{(e,\nu)}LSU_2 = \Gamma_{\mathfrak{p}_\alpha}(e,\nu) \bigoplus \mathfrak{Y}(e,\nu).$$

Proposition 4.19. $\mathfrak{Y}(e,\nu)$, *the subspace in* $\mathbb{R}^4 \bigoplus (\mathbb{R}^4)^3$, *consists of of matrices*

$$\{(0,Y) := \begin{bmatrix} 0 & \\ & Y \end{bmatrix}_{4\times 4} | Y \in \mathfrak{gl}(3,\mathbb{R})\}. \tag{4.2.3}$$

Proof. Recall that the orbit map $\sigma_u : a \mapsto u \cdot a$ of the structural group $GL_3(\mathbb{R})$ action $\sigma : LSU_2 \times GL_3(\mathbb{R}) \to LSU_2$, induces $\sigma(e, I_3)_{*,e} : \mathfrak{gl}(3,\mathbb{R}) \to T_{(e,I_3)}(SU_2 \times GL_3(\mathbb{R})) := \mathfrak{Y}(e, I_3)$, a linear isomorphism $\mathfrak{Y}(e, I_3) \simeq \mathfrak{gl}(3,\mathbb{R})$. For each $Y \in \mathfrak{gl}(3,\mathbb{R})$, therefore,

$$\sigma_{u*,e} Y = \tfrac{d}{dt}|_{t=0}(e, I_3) \cdot \exp_{GL_3(\mathbb{R})} tY = \tfrac{d}{dt}|_{t=0}(e, \exp_{GL_3(\mathbb{R})} tY) = (0, Y),$$

as the structure group action is fibre-transitive. □

Let S be a connected matrix Lie group, on which $S \times S$ naturally acts by (4.0.1). Let $(x, \nu) \in LS \simeq S \times GL(\mathfrak{s})$ and $f \in S \times S$. Formula (2.4.1) gives $L_f(x, \nu) = (f \cdot x, (L_f)_{*,e}\nu)$. If $f = (a, b)$ and $x = e$, then $f \cdot x = (a, b) \cdot x = axb^{-1}$, whereas the natural lift $(L_f)_{*,x}\nu = \frac{d}{dt}|_{t=0} L_f \circ \exp_S t\nu = \frac{d}{dt}|_{t=0} a(\exp_S t\nu)b^{-1} = a\nu b^{-1}$, so that K acts on LM by

$$L^f(x, u) = a(x, u)b^{-1}. \tag{4.2.4}$$

Proposition 4.20. *The horizontal subspace of the Cartan-Schouten connection $\Gamma(\mathfrak{p}_\alpha)$ on the $GL_3(\mathbb{R})$-principal bundle $SU_2 \times GL_3(\mathbb{R}) \to SU_2$ is given by $\Gamma_{\mathfrak{p}_\alpha}(e, I_3) = \mathrm{span}_{\mathbb{R}}\{{}^\sharp X_i^\alpha : 1 \le i \le 3\}$ in the space $\mathbb{R}^4 \bigoplus (\mathbb{R}^4)^3$, where*

$$\begin{aligned}
{}^\sharp X_1^\alpha &= U_1 \oplus (-I_2, -\alpha U_3, \alpha U_2),\\
{}^\sharp X_2^\alpha &= U_2 \oplus (\alpha U_3, -I_2, -\alpha U_1),\\
{}^\sharp X_3^\alpha &= U_3 \oplus (-\alpha U_2, \alpha U_1, -I_2).
\end{aligned}$$

Proof. ${}^\sharp X_i^\alpha = \frac{d}{dt}|_{t=0} {}^\sharp K_i^\alpha(t)$, where ${}^\sharp K_i^\alpha = L^{K_i^\alpha}(e, \nu)$ is the induced curve of K_i^α in LS for $1 \le i \le 3$. Formula 4.2.4 shows that ${}^\sharp K_1^\alpha = (\epsilon_{r_\alpha t}\epsilon_{s_\alpha t}^{-1}, \epsilon_{r_\alpha t}\nu\epsilon_{s_\alpha t}^{-1})$, ${}^\sharp K_2^\alpha = (p\epsilon_{r_\alpha t}\epsilon_{s_\alpha t}^{-1}p^{-1}, p\epsilon_{r_\alpha t}p^{-1}\nu p\epsilon_{s_\alpha t}^{-1}p^{-1})$, ${}^\sharp K_3^\alpha = (q\epsilon_{r_\alpha t}\epsilon_{s_\alpha t}^{-1}q^{-1}, q\epsilon_{r_\alpha t}q^{-1}\nu q\epsilon_{s_\alpha t}^{-1}q^{-1})$. Proposition 4.19 then gives ${}^\sharp K_i^\alpha = (c_j, (a_{1j}, a_{2j}, a_{3j}))$ for $1 \le i, j \le 3$; precisely, we have

$$\begin{aligned}
c_1(t) &= \cos(r_\alpha - s_\alpha)tI + \sin(r_\alpha - s_\alpha)tU_1, a_{11}(t) = -\sin(r_\alpha - s_\alpha)tI + \cos(r_\alpha - s_\alpha)tU_1,\\
a_{12}(t) &= \cos(r_\alpha + s_\alpha)tU_2 - \sin(r_\alpha + s_\alpha)tU_3, a_{13}(t) = \sin(r_\alpha + s_\alpha)tU_2 + \cos(r_\alpha + s_\alpha)tU_3,\\
c_2(t) &= \cos(r_\alpha - s_\alpha)tI + \sin(r_\alpha - s_\alpha)tU_2, a_{21}(t) = \cos(r_\alpha + s_\alpha)tU_1 + \sin(r_\alpha + s_\alpha)tU_3,\\
a_{22}(t) &= -\sin(r_\alpha - s_\alpha)tI + \cos(r_\alpha - s_\alpha)tU_2, a_{23}(t) = -\sin(r_\alpha + s_\alpha)tU_1 + \cos(r_\alpha + s_\alpha)tU_3,\\
c_3(t) &= \cos(r_\alpha - s_\alpha)tI + \sin(r_\alpha - s_\alpha)tU_3, a_{31}(t) = \cos(r_\alpha + s_\alpha)tU_1 - \sin(r_\alpha + s_\alpha)tU_2,\\
a_{32}(t) &= \sin(r_\alpha + s_\alpha)tU_1 + \cos(r_\alpha + s_\alpha)tU_2, a_{33}(t) = -\sin(r_\alpha - s_\alpha)tI + \cos(r_\alpha - s_\alpha)tU_3,
\end{aligned}$$

by elementary matrix calculations. It hence follows that

$$(r_\alpha - s_\alpha)U_1 \oplus (-(r_\alpha - s_\alpha)I_2, -(r_\alpha + s_\alpha)U_3, (r_\alpha + s_\alpha)U_2),$$

$$(r_\alpha - s_\alpha)U_2 \oplus ((r_\alpha + s_\alpha)U_3, -(r_\alpha - s_\alpha)I_2, -(r_\alpha + s_\alpha)U_1),$$

$$(r_\alpha - s_\alpha)U_3 \oplus (-(r_\alpha + s_\alpha)U_2, (r_\alpha + s_\alpha)U_1, -(r_\alpha - s_\alpha)I_2),$$

equal $\frac{d}{dt}|_{t=0}{}^\sharp K_1^\alpha$, $\frac{d}{dt}|_{t=0}{}^\sharp K_2^\alpha$, and $\frac{d}{dt}|_{t=0}{}^\sharp K_3^\alpha$, respectively. □

We compute the horizontal lift equation of an arbitrary curve $c : [0,1] \to SU_2$. We represent c as $c_0 I_2 + c_1 U_1 + c_2 U_2 + c_3 U_3$. The curve $(c, e) : t \mapsto (c(t), e) \subset SU_2 \times SU_2$ then gives rise to a lift of c to $SU_2 \times SU_2$; moreover, it naturally induces the lift $\gamma = L^{(c,e)}(e, I_3)$ of c to $SU_2 \times GL_3(\mathbb{R})$, whence

$$\gamma = \begin{bmatrix} c_0 & -c_1 & -c_2 & -c_3 \\ c_1 & c_0 & c_3 & -c_2 \\ c_2 & -c_3 & c_0 & c_1 \\ c_3 & c_2 & -c_1 & c_0 \end{bmatrix}, \quad \dot\gamma = \begin{bmatrix} \dot c_0 & -\dot c_1 & -\dot c_2 & -\dot c_3 \\ \dot c_1 & \dot c_0 & \dot c_3 & -\dot c_2 \\ \dot c_2 & -\dot c_3 & \dot c_0 & \dot c_1 \\ \dot c_3 & \dot c_2 & -\dot c_1 & \dot c_0 \end{bmatrix}. \tag{4.2.5}$$

Our next step is to work out $(L^{(c,e)}(e, I_3))^{-1}\dot\gamma$ that appears in the horizontal lift equation in Proposition 4.18. By straightforward computation

$$(L^{(c,e)}(e, I_3))^{-1}\dot\gamma = \begin{bmatrix} & l & m & n \\ -l & & -n & m \\ -m & n & & -l \\ -n & -m & l & \end{bmatrix},$$

where the time-dependent entries are

$$l = -c_0\dot c_1 + c_1\dot c_0 - c_2\dot c_3 + c_3\dot c_2,$$

$$m = -c_0\dot c_2 + c_1\dot c_3 + c_2\dot c_0 - c_3\dot c_1,$$

$$n = -c_0\dot c_3 - c_1\dot c_2 + c_2\dot c_1 + c_3\dot c_0.$$

Denote by ω_α the connection form of the Cartan-Schouten connection $\Gamma_{\mathfrak{p}_\alpha}$ in $SU_2 \times GL_3(\mathbb{R}) \to GL_3(\mathbb{R})$. For each vector field X on $SU_2 \times GL_3(\mathbb{R})$, let $X^{\perp_\alpha}$ be its vertical component with respect to ω_α. The next step is to compute the vertical vector $\xi^{\perp_\alpha}$. For that purpose, we need real coefficients a_i^α ($1 \le i \le 3$, possibly depending on α), for which

$$(L^{(c,e)}(e, I_3))^{-1}\dot\gamma - \sum_{i=1}^{3} a_i^{\alpha\sharp} X_i^\alpha \in \mathfrak{Y}(e, \nu).$$

Simple computation then shows $a_1 = -l(0)$, $a_2 = -m(0)$, and $a_3 = -n(0)$. Now

$$\begin{aligned} {}^\sharp X_1^\alpha &= E_{21} - E_{12} + \alpha(E_{34} - E_{43}), \\ {}^\sharp X_2^\alpha &= E_{31} - E_{13} + \alpha(E_{42} - E_{24}), \\ {}^\sharp X_3^\alpha &= E_{41} - E_{14} + \alpha(E_{23} - E_{32}), \end{aligned}$$

with respect to the basis ν. Since $\omega_\alpha(e,\nu) : \mathfrak{Y}(e,\nu) \simeq \mathfrak{gl}(3,\mathbb{R})$ is a linear isomorphism, it follows immediately from (4.2.3) that

$$\omega_\alpha(e,\nu)(L^{c^{-1}}\dot{\gamma})^{\perp_\alpha} = (1-\alpha)\begin{bmatrix} & -n & m \\ n & & -l \\ -m & l & \end{bmatrix}.$$

Recall that, under the matrix model (4.2.2), $c = c_0 I_2 + c_1 U_1 + c_2 U_2 + c_3 U_3$. Hence

$$c^{-1}\dot{c} = \begin{bmatrix} -il & m - in \\ -m - in & il \end{bmatrix} = -lU_1 - mU_2 - nU_3.$$

But,

$$\begin{aligned} \mathrm{ad}_{\mathfrak{s}}(c^{-1}\dot{c})U_1 &= -m[U_2, U_1] - n[U_3, U_1] = 2nU_2 - 2mU_3, \\ \mathrm{ad}_{\mathfrak{s}}(c^{-1}\dot{c})U_2 &= -l[U_1, U_2] - n[U_3, U_2] = -2nU_1 + 2lU_3, \\ \mathrm{ad}_{\mathfrak{s}}(c^{-1}\dot{c})U_3 &= -m[U_1, U_3] - m[U_2, U_3] = 2mU_1 - 2lU_2. \end{aligned}$$

Therefore the matrix of $\mathrm{ad}_{\mathfrak{s}}(c^{-1}\dot{c})$ in the basis ν is

$$\mathrm{ad}_{\mathfrak{s}}(c^{-1}\dot{c}) = \frac{1}{2}\begin{bmatrix} & -n & m \\ n & & -l \\ -m & l & \end{bmatrix}.$$

In conclusion, the horizontal lift equation for ω_α with respect to γ is

$$\dot{a}a^{-1} = \frac{\alpha - 1}{2}\mathrm{ad}_{\mathfrak{s}}(c^{-1}\dot{c}).$$

Let S be a connected Lie group.

4.2.3 Cartan-Schouten connections for connected Lie groups

The notation in Section 2.1 is in effect. Fix a frame ν in $\mathfrak{s}$. As in Example 2.5 we can identify the frame bundle on S with the bundle

$$\pi : S \times GL(\mathfrak{s}) \to S, (s, a) \mapsto s,$$

with which we will work instead. Denote $u_e = (e, \mathrm{id}_{\mathfrak{s}}) \in S \times GL(\mathfrak{s})$ which corresponds to the point $(e, \nu) \in LS$.

Proposition 4.21. *Let $X \in \mathfrak{s}$. Its push forward through section $S \mapsto S \times GL(\mathfrak{s}), s \mapsto (s, \mathrm{id}_{\mathfrak{s}})$ will still be denoted by X. Then $X^{\perp_\alpha} = {}^\sharp(\pi_{*,e}X, 0)_{\mathfrak{h}}$, the $\mathfrak{h}$-component being taken with respect to the decomposition $\mathfrak{s} \bigoplus \mathfrak{s} = \mathfrak{h} \bigoplus \mathfrak{p}_\alpha$.*

Proof. We first observe that $X = {}^\sharp(\pi_{*,e}X, 0) \in \mathfrak{s} \bigoplus \{0\} \subset \mathfrak{s} \bigoplus \mathfrak{s}$. Indeed, $\pi_{*,(e,I_3)}X \in T_eS \simeq \mathfrak{s}$, so that $(\pi_{*,(e,I_3)}X, 0) \in \mathfrak{s} \bigoplus \{0\} \subset \mathfrak{s} \bigoplus \mathfrak{s}$. But the commutivity of the bundle projection with $\sharp$ and $\flat$ (recalling Remark 2.74) gives

$$\begin{aligned}
\pi_{*,(e,I_3)}{}^\sharp(\pi_{*,(e,I_3)}X, 0) &= {}^\flat(\pi_{*,(e,I_3)}X, 0) \\
&= \frac{d}{dt}|_{t=0} \exp_{S\times S} t(\pi_{*,(e,I_3)}X, 0) \cdot (e, I_3) \\
&= \frac{d}{dt}|_{t=0} \exp_S t\pi_{*,(e,I_3)}X \cdot e = \pi_{*,(e,I_3)}X.
\end{aligned}$$

The claim follows as $\pi_{*,u_e} : T_{u_e}(S \times \mathrm{id}_{\mathfrak{s}}) \to \mathfrak{s}$ is one-to-one.

Next, consider the direct sum depomposition $(X, 0) = (X, 0)_{\mathfrak{h}} \oplus (X, 0)_{\mathfrak{p}_\alpha}$ with respect to $\mathfrak{s} \bigoplus \mathfrak{s} = \mathfrak{h} \bigoplus \mathfrak{p}_\alpha$. Proposition 2.90 asserts that ${}^\sharp(X, 0)_{\mathfrak{p}_\alpha} \in \Gamma(\mathfrak{p}_\alpha)(e, I_3)$; moreover, the integral curve of ${}^\sharp(X, 0)_{\mathfrak{h}}$ through $(e, I_3) \in S \times GL(\mathfrak{s})$ satisfies

$$\exp t^\sharp(X, 0)_{\mathfrak{h}} = \exp_{S\times S} t^\sharp(X, 0)_{\mathfrak{h}} \cdot (e, \nu)$$

for all t since $\sharp(e, I_3) : f \mapsto (f, (L_f)_{*,e})$ is a bundle map. Hence,

$$\pi^\sharp_{*,(e,I_3)}(X, 0)_{\mathfrak{h}} = \frac{d}{dt}|_{t=0}(\exp_{S\times S} t^\sharp(X, 0)_{\mathfrak{h}} \cdot (e, I_3)) = \frac{d}{dt}|_{t=0} \exp_S t^\sharp(X, 0)_{\mathfrak{h}} \cdot e = 0,$$

as $\exp_S t^\sharp(X, 0)_{\mathfrak{h}} \in H$ for all t: ${}^\sharp(X, 0)_{\mathfrak{h}}$ is a vertical vector at (e, I_3) as desired. $\square$

Theorem 4.22. *Let $c : [0,1] \to S$ be a piecewise C^1 curve on the Lie group S and choose $u_0 \in LS$ with $\pi(u_0) = c(0)$. Then the horizontal lift equation of the Cartan-Schouten connection $\Gamma_{\mathfrak{p}_\alpha}$ for the reference curve $\gamma = L^{(c,e)}(u_0)$ is*

$$\dot{a}a^{-1} = \frac{\alpha - 1}{2}\mathrm{ad}_{\mathfrak{s}}(c^{-1}\dot{c}).$$

Proof. Let $u_0 = (e, \nu)$. As in Example 2.5 we can identify the linear frame bundle with the bundle $S \times GL(\mathfrak{s}) \to S$ such that $(e, \mathrm{id}_{\mathfrak{s}})$ corresponds to u_0. Therefore we can work with the bundle $S \times GL(\mathfrak{s}) \to S$ instead of the linear frame bundle.

We identify c with the curve $(c, e) \subset S \times S$ and choose the reference curve $\gamma = L^{(c,e)}(u_0)$ in $S \times GL(\mathfrak{s})$. This shows that Proposition 4.6 is indeed applicable, from which it follows $L^{(c,e)}(e, \mathrm{id}_{\mathfrak{s}})^{-1}\dot{\gamma} = (c^{-1}\dot{c}; 0)$.

Denote $Y = c^{-1}\dot{c}$. Lemma 4.21 shows $(L^{(c,e)}(e, \mathrm{id}_{\mathfrak{s}})^{-1}\dot{\gamma})^{\perp_\alpha} = {}^\sharp(Y, 0)_{\mathfrak{h}}$. Just as in Theorem 4.6, we obtain $(Y, 0)_{\mathfrak{h}} = \frac{1-\alpha}{2}(Y, Y)$. Theorem 2.77 then gives

$$\omega^\alpha(u_0)Y^{\perp_\alpha} = \Lambda_\alpha^\sharp(Y, 0)_{\mathfrak{h}} = \mathrm{ad}_{\mathfrak{h}}(Y, 0)_{\mathfrak{h}} = \frac{1-\alpha}{2}\mathrm{ad}_{\mathfrak{s}}Y$$

via the identification $\mathfrak{h} = \Delta(\mathfrak{s} \bigoplus \mathfrak{s}) \simeq \mathfrak{s}$. The conclusion then follows readily by Proposition 4.18. $\square$

We conclude this section with some comments on the horizontal lift equation and Lie's third fundamental theorem. First, remark that if $c : [0,1] \to S$ is a piecewise C^1 curve on S, then

$$c^{-1}\dot{c} : [0,1] \to \mathfrak{s}$$

is a C^1-path in the Lie algebra, and

$$a : [0,1] \to GL(\mathfrak{s}).$$

Endow S with a scalar product. Duistermaat and Kolk [7, Section 1.14] proved the following results. The space $\mathcal{P}(\mathfrak{s})$ of continuous paths $[0,1] \to \mathfrak{s}$, equipped with the supremum norm, is a Banach space. For $\gamma \in \mathcal{P}(\mathfrak{s})$, let $a_\gamma : [0,1] \to GL(\mathfrak{s})$ determined by the ODE

$$\dot{a_\gamma}(t)a_\gamma(t) = \mathrm{ad}_{\mathfrak{s}}(\gamma(t)), \quad a_\gamma(0) = \mathrm{id}_{\mathfrak{s}}. \tag{4.2.6}$$

Then, the multiplication

$$(\gamma \cdot \delta)(t) = \gamma(t) + a_\gamma(t)(\delta(t))$$

endows $\mathcal{P}(\mathfrak{s})$ with the structure of Banach Lie groups. They also identify a closed, connected, normal Banach Lie subgroup $\mathcal{P}(\mathfrak{s})_0$ of $\mathcal{P}(\mathfrak{s})$ and show that the quotient $\mathcal{P}(\mathfrak{s})/\mathcal{P}(\mathfrak{s})_0$ is a simply-connected Lie group with Lie algebra $\mathfrak{s}$. This construction provides a direct, natural solution to Lie's third fundamental theorem (in global form).

We remark that (4.2.6) is precisely the horizontal lift equation for $\gamma = c^{-1}\dot{c}$ and $\alpha = 3$. It should be interesting to explore the structure on $\mathcal{P}(\mathfrak{s})$ induced by the horizontal lift equation for an arbitrary α.

4.2.4 Holonomy groups of Cartan-Schouten connections.

We give a discussion of the holonomy groups of Cartan-Schouten connections $\Gamma_{\mathfrak{p}_\alpha}$, $\alpha \in \mathbb{R}$. Results are obtained at the Lie algebra level. We first work on the special case where the underlying space is the special unitary group SU_2. Results are then generalized, by the entire same methods, to arbitrary Lie groups.

Lemma 4.23. *Let S be a Lie group and $\mathfrak{s}$ its Lie algebra. Then*

$$[\mathfrak{p}_\alpha, \mathfrak{p}_\alpha]_{\mathfrak{h}} = \frac{1-\alpha^2}{4}[\mathfrak{h}, \mathfrak{h}].$$

Proof. Let $X, Y \in \mathfrak{s}$. Then $[(r_\alpha X, s_\alpha X), (r_\alpha Y, s_\alpha Y)] = (r_\alpha^2[X,Y], s_\alpha^2[X,Y])$. Set $[(r_\alpha X, s_\alpha X), (r_\alpha Y, s_\alpha Y)]_{\mathfrak{h}} = (W, W)$ for $W \in \mathfrak{s}$. Then

$$\tfrac{1}{r_\alpha}(r_\alpha^2[X,Y] - W) = \tfrac{1}{s_\alpha}(s_\alpha^2[X,Y] - W),$$

so that $W = -r_\alpha s_\alpha[X,Y]$, where $r_\alpha s_\alpha = \frac{1}{4}(\alpha^2 - 1)$, verifying the conclusion. □

Proposition 4.24. *Let S be a connected, simple Lie group and $u_e = (e, \nu) \in LS$. The Lie algebra* $\mathrm{Lie}\Phi^\alpha(u_e)$ *of the holonomy group* $\Phi^\alpha(u_e)$ *of the Cartan-Schouten connections $\Gamma_{\mathfrak{p}_\alpha}$ on SU_2 is isomorphic to $\mathfrak{s}$ if $\alpha \neq \pm 1$, and it is trivial if $\alpha = \pm 1$.*

Proof. As $\Gamma_{\mathfrak{p}_\alpha}$ is a canonical connection, Proposition 2.91(c) states that

$$\mathrm{Lie}\Phi^\alpha(u_e) = \mathrm{span}_{\mathbb{R}}\mathrm{ad}[\mathfrak{p}_\alpha, \mathfrak{p}_\alpha]_{\mathfrak{h}}.$$

By Lemma 4.23, $[\mathfrak{p}_\alpha, \mathfrak{p}_\alpha]_{\mathfrak{h}} = \frac{1-\alpha^2}{4}[\mathfrak{h}, \mathfrak{h}]$. Since S is simple, $\mathfrak{h} \simeq \mathfrak{s}$ and $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$. If $\alpha = \pm 1$, then $[\mathfrak{p}_\alpha, \mathfrak{p}_\alpha]_{\mathfrak{h}} = 0$ which implies our claim. If $\alpha \neq \pm 1$, then $\mathrm{Lie}\Phi^\alpha(u_e) = \mathrm{span}_{\mathbb{R}}\mathrm{ad}(\mathfrak{h}) \subset \mathfrak{gl}(\mathfrak{s} \bigoplus \mathfrak{s})$. Remark that for $(X, X) \in \mathfrak{h}$, $(Y, Z) \in \mathfrak{s} \bigoplus \mathfrak{s}$,

$$\mathrm{ad}(X, X)(Y, Z) = (\mathrm{ad}(X)Y, \mathrm{ad}(X)Z).$$

So $\mathrm{span}_{\mathbb{R}}\mathrm{ad}(\mathfrak{h}) \subset \mathfrak{gl}(\mathfrak{s} \bigoplus \mathfrak{s})$ equals the image of the map

$$\mathrm{ad} \oplus \mathrm{ad} : \mathfrak{gl}(\mathfrak{s}) \bigoplus \mathfrak{gl}(\mathfrak{s}) \subset \mathfrak{gl}(\mathfrak{s} \bigoplus \mathfrak{s}),$$

which is the diagonal embedding of $\mathrm{ad}_{\mathfrak{s}}(\mathfrak{s})$ into $\mathfrak{gl}(\mathfrak{s} \bigoplus \mathfrak{s})$. As S is simple, $\mathrm{ad}_{\mathfrak{s}}(\mathfrak{s})$ is a Lie algebra isomorphic to $\mathfrak{s}$. Therefore, if $\alpha \neq \pm 1$, $\mathrm{Lie}^\alpha(u_e)$ is isomorphic to $\mathfrak{s}$. $\square$

5.0 THE DIRAC OPERATOR

In this chapter we study the Dirac operators associated to Cartan-Schouten connections. First, we give a brief introduction to concepts related to Dirac operators. Our main reference is [17].

5.1 PRELIMINARIES

Let F be a field, let V be an F-vector space, and let q be a non-degenerate quadratic form on V. The tensor algebra of V is denoted by $TV = \sum_{r\geq 0} \bigotimes^r V$.

5.1.1 Clifford algebras.

Definition 5.1. The *Clifford algebra* of the quadratic space (V, q) is the quotient

$$\mathrm{C}\ell(V, q) = TV/I_q(V),$$

where $I_q(V)$ is the ideal in TV generated by all elements $v \otimes v + q(v)\mathsf{1}$ for $v \in V$.

There exists a canonical embedding

$$\iota : V \to \mathrm{C}\ell(V, q).$$

The algebra $\mathrm{C}\ell(V, q)$ is generated by $V \subset \mathrm{C}\ell(V, q)$ subject to the relations

$$v \cdot v = -q(v)\mathsf{1}, \quad \text{for } v \in V,$$

which can be used to give the following universality property of $\mathrm{C}\ell(V, q)$.

Proposition 5.2. [17, I, Proposition 1.1] *Let $f : V \to \mathcal{A}$ be a linear map into an associative F-algebra with unit, such that $f(v) \cdot f(v) = -q(v)1$ for all $v \in V$. Then f extends uniquely to an F-algebra homomorphism $\widetilde{f} : \mathrm{C}\ell(V, q) \to \mathcal{A}$. Furthermore, up to isomorphism, $\mathrm{C}\ell(V, q)$ is the unique associative F-algebra with this property.*

Corollary 5.3. *If ρ is an orthogonal transformation on (V, q), then there exists a unique algebra map $c(\rho)$ that makes the diagram*

$$\begin{array}{ccc} (V,q) & \xrightarrow{\rho} & (V,q) \\ {\scriptstyle\iota}\big\downarrow & & \big\downarrow{\scriptstyle\iota} \\ \mathrm{C}\ell(V,q) & \xrightarrow{c(\rho)} & \mathrm{C}\ell(V,q) \end{array}$$

commutative.

Remark 5.4.

(1) As an application of Corollary 5.3, consider the automorphism

$$c(\alpha) : \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V, q)$$

which extends the map $\alpha(v) = -v$ on V. Since $\alpha^2 = 1$, there exists a decomposition

$$\mathrm{C}\ell(V, q) = \mathrm{C}\ell^0(V, q) \bigoplus \mathrm{C}\ell^1(V, q)$$

into the two eigenspaces of $c(\alpha)$, i.e. $\phi \in \mathrm{C}\ell^i(V, q)$ if and only if $c(\alpha)\phi = (-1)^i\phi$ for $i = 0$ or 1 . The 1-eigenspace $\mathrm{C}\ell^0(V, q)$ is a subalgebra of $\mathrm{C}\ell(V, q)$ and is called the *even part* of $\mathrm{C}\ell(V, q)$; $\mathrm{C}\ell^1(V, q)$ is called the *odd part*.

(2) Let G be a Lie group. Corollary 5.3 asserts that every orthogonal representation $\phi : G \to O(\mathbb{R}^n)$ lifts to a representation $c(\phi) : G \to \mathrm{End}(\mathrm{C}\ell(n))$ on the Clifford algebra.

Notation. We denote by $\mathrm{C}\ell(n)$ the Clifford algebra $\mathrm{C}\ell(V, q)$ for $V = \mathbb{R}^n$ and $q(x) = |x|^2$. The complexification of the algebra $\mathrm{C}\ell(n)$ is just the Clifford algebra (over $\mathbb{C}$) corresponding to the complexified quadratic form, i.e. $\mathrm{C}\ell(n) \bigotimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{C}\ell(\mathbb{C}^n, q \otimes \mathbb{C})$. However, all non-degenerate quadratic forms on $\mathbb{C}^n$ are equivalent over $\mathbb{C}$. Hence, set $q_{\mathbb{C}}(z) = z_1^2 + \cdots + z_n^2$ and define $\mathbb{C}\ell(n) = \mathrm{C}\ell(\mathbb{C}^n, q_{\mathbb{C}})$. Then,

$$\mathbb{C}\ell(n) \simeq \mathrm{C}\ell(n) \bigotimes_{\mathbb{R}} \mathbb{C}.$$

We now consider the *multiplicative group of units* $C\ell^\times(V,q)$ in the Clifford algebra $C\ell(V,q)$. This is a Lie group of dimension 2^n when $\dim V = n < \infty$ and F is either $\mathbb{R}$ or $\mathbb{C}$; in general, its associated Lie algebra is $\mathfrak{cl}^\times(V,q) = C\ell(V,q)$ with Lie bracket given by $[x,y] = xy - yx$. The group of units acts on the Clifford algebra by the *adjoint representation*

$$\mathrm{Ad}_{C\ell} : C\ell^\times(V,q) \to GL(C\ell(V,q)), \quad \mathrm{Ad}_{C\ell}(y)x = yxy^{-1}.$$

The induced Lie algebra representation is given by

$$\mathrm{ad}_{C\ell} : \mathfrak{cl}^\times(V,q) \to \mathrm{Der}(C\ell(V,q)), \quad \mathrm{ad}_{C\ell}(y)x = [y,x].$$

For later purposes we also consider the *twisted adjoint representation*, defined by

$$\widetilde{\mathrm{Ad}} : C\ell^\times(V,q) \to GL(C\ell(V,q)), \quad \widetilde{\mathrm{Ad}}(\phi)(y) = c(\alpha)(\phi)y\phi^{-1}.$$

5.1.2 Spin groups.

For $n \geq 1$, denote by $M_n(\mathsf{F})$ the algebra of $n \times n$ matrices with entries in F. We denote by $P(V,q)$ the subgroup of $C\ell^\times(V,q)$ generated by the elements $v \in V$ with $q(v) \neq 0$. The group $P(V,q)$ has certain important subgroups.

Definition 5.5. The *pin group* of (V,q) is the subgroup $\mathrm{Pin}(V,q)$ of $P(V,q)$ generated by the elements $v \in V$ with $q(v) = \pm 1$. The associated *spin group* of (V,q) is defined by $\mathrm{Spin}(V,q) = \mathrm{Pin}(V,q) \cap C\ell^0(V,q)$.

Theorem 5.6. [17, I, Theorem 2.9] *Let V be a finite-dimensional F-vector space with $\mathsf{F} = \mathbb{R}$ or $\mathbb{C}$. Then, there is a short exact sequence*

$$0 \to \mathsf{F} \to \mathrm{Spin}(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} SO(V,q) \to 1.$$

For $C\ell(n)$ we obtain the following

Theorem 5.7. [11, 17, I, Theorem 2.10] *For all $n \geq 3$, SO_n is connected and $\pi_1 SO_n \simeq \mathbb{Z}_2$. Moreover, there is a short exact sequence*

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}_n \xrightarrow{\widetilde{\mathrm{Ad}}} SO_n \to 1,$$

and $\widetilde{\mathrm{Ad}} : \mathrm{Spin}_n \to SO_n$ is the universal covering of SO_n.

5.1.3 Spin representations

The algebra $\mathbb{C}\ell(2k)$ is isomorphic to $M_{2^k}(\mathbb{C})$. The algebra $\mathbb{C}\ell(2k+1)$ is isomorphic to $M_{2^k}(\mathbb{C}) \bigoplus M_{2^k}(\mathbb{C})$. Therefore, $\mathbb{C}\ell(2k)$ has a unique simple module. Its $\mathbb{C}$-dimension is 2^k. The algebra $\mathbb{C}\ell(2k+1)$ has two simple modules, each of dimension 2^k. By restricting the action of $\mathbb{C}\ell(n)$ on a simple module to $\mathrm{Spin}_n \subset C\ell(n) \subset \mathbb{C}\ell(n)$ we obtain a representation of Spin_n, called *complex spinor representation*. Note that if $\mathbb{C}\ell(n)$ has two simple modules, the Spin_n representations obtained by restriction are isomorphic.

Notation. We will denote by $(\underline{\pi}, W)$ the $\mathbb{C}$-spin representation of Spin_n. Its dimension is $2^{[n/2]}$.

5.1.4 Spin structures.

We consider an oriented Riemannian manifold M. Recall from Example 2.36 that the linear frame bundle LM may be reduced to a principal SO_n-subbundle O^+M, which consists of positively-oriented linear frames on M.

Definition 5.8. Let $n \geq 3$. Then a *Spin structure* on M is a principal Spin_n-bundle $P(M, \mathrm{Spin}_n)$ together with a two-sheeted covering

$$\xi : P(M, \mathrm{Spin}_n) \to O^+M,$$

such that $\xi(p \cdot a) = \xi(p) \cdot \widetilde{\mathrm{Ad}}(a)$ for all $p \in P(M, \mathrm{Spin}_n)$ and $a \in \mathrm{Spin}_n$.

Remark 5.9. The existence of a Spin structure on M is equivalent to the vanishing of the second Stiefel-Whitney class of M.

We consider the standard representation $\mathrm{id} : SO_n \to SO(\mathbb{R}^n)$ of the special orthogonal group in $\mathbb{R}^n$. This gives rise to tangent bundle $TM = O^+M \times_{\mathrm{id}} \mathbb{R}^n$; moreover, it lifts (recalling Remark 5.4) to the representation $c(\mathrm{id}) : SO_n \to \mathrm{End}(C\ell(n))$.

Definition 5.10. Let M be an oriented Riemannian manifold. The *Clifford bundle* associated to the tangent bundle TM is defined by

$$C\ell(TM) = O^+M \times_{c(\mathrm{id})} C\ell(n).$$

A fibre bundle $\mathfrak{S}$ over M is called a *bundle of left modules* over $C\ell(TM)$ if at each point $x \in M$, the fibre $\mathfrak{S}_x$ is a left module over the Clifford algebra $C\ell(TM)_x$.

It is natural to look for bundles of irreducible modules over the Clifford bundle $C\ell(TM)$. Such bundles can be constructed if a Spin structure on M exists.

Definition 5.11. Let M be an orientable Riemannian manifold with a Spin structure $\xi : P(M, \mathrm{Spin}_n) \to O^+M$. A *complex spinor bundle* over M is defined by

$$\mathfrak{S}_{\mathbb{C}} = P(M, \mathrm{Spin}_n) \times_\mu U,$$

where U is a complex left module of $C\ell(n)$ and μ is the restriction to Spin_n of the $C\ell(n)$ action on U.

5.1.5 Dirac operators.

Let M be an orientable Riemannian manifold with Clifford bundle $C\ell(TM)$; let $\mathfrak{S}$ be any bundle of left modules over $C\ell(TM)$. Denote by $C^\infty(M, \mathfrak{S})$ the space of smooth sections in $\mathfrak{S}$. Let Γ be an affine metric connection on M. By Proposition 2.37, Γ induces a connection $\widetilde{\Gamma}$ on $P(M, \mathrm{Spin}_n)$ and further, a covariant derivative $\widetilde{\nabla}$ on $\mathfrak{S}$.

Definition 5.12. We define a first-order differential operator

$$D : C^\infty(M, \mathfrak{S}) \to C^\infty(M, \mathfrak{S}),$$

called the *Dirac operator* of ∇, by setting

$$Df = \textstyle\sum_j e_j \cdot \widetilde{\nabla}_{e_j} f$$

at $x \in M$, where e_j $(1 \leq j \leq n)$ is an orthonormal basis of T_xM and where $\cdot$ denotes the Clifford algebra action.

5.2 DIRAC OPERATORS ON REDUCTIVE HOMOGENEOUS SPACES

In this section we shall consider the special case where $M \simeq K/H$ is a reductive homogeneous space, with the $\mathrm{Ad}|_H$-invariant decomposition $\mathfrak{k} = \mathfrak{h} \bigoplus \mathfrak{m}$. We shall fix the origin $o \in M$ to be the point with stabilizer H. In order to study Dirac operators, it is necessary to have a Spin structure on M, which then depends on a choice of Riemannian metric.

Definition 5.13. A Riemannian metric g on a K-homogeneous manifold M is K-*invariant* if it is invariant under the group action of K.

Notation. Let M be a K-homogeneous manifold and g a Riemannian metric on M. The set of linear metric connections is denoted by $\mathrm{Conn}(M, g)$. If g is K-invariant, then we denote the set of K-invariant connections by $\mathrm{Conn}_K(M, g)$. The subset in $\mathrm{Conn}_K(M, g)$ that consists of canonical connections is denoted by $\mathrm{Can}_K(M, g)$.

Proposition 5.14. [13, vol. II, X, Corollary 3.2] *There is a natural one-to-one correspondence between the K-invariant indefinite Riemannian metrics g on M and the $\mathrm{Ad}|_H$-invariant non-degenerate symmetric bilinear forms $\langle\ ,\ \rangle$ on $\mathfrak{m}$, given by*

$$\langle X, Y\rangle = g(X, Y)_o.$$

Next, we set the identification $\mathrm{Conn}_K(M) \leftrightarrow \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$ described in Theorem 2.83. The correspondence (3.1.1) indicates that if $\eta \in \mathrm{Hom}_H(\mathfrak{m} \bigotimes \mathfrak{m}, \mathfrak{m})$ represents $\Lambda \in \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$, the connection map of the covariant derivative ∇^η, then $\eta(X, Y) = \Lambda(X)Y$. Denote by $\mathfrak{so}(\mathfrak{m}, g_o)$ the set of skew-symmetric transformation in $\mathfrak{m}$ with respect to the bilinear form g_o.

Proposition 5.15. *Let Γ be a K-invariant connection whose connection map is Λ. Let g be a K-invariant Riemannian metric on M. Then $\Gamma \in \mathrm{Conn}_K(M, g)$ if and only if $\Lambda\mathfrak{m} \subset \mathfrak{so}(\mathfrak{m}, g_o)$.*

Proof. We have

$$\begin{aligned}\nabla g(X, Y; Z) &= Z \cdot g(X, Y) - g(\nabla^\eta_Z X, Y) - g(X, \nabla^\eta_Z Y)\\ &= \nabla^0 g(X, Y; Z) - g(\Lambda(Z)X, Y) - g(X, \Lambda(Z)Y)\end{aligned}$$

for all $X, Y, Z \in \mathfrak{m}$. Note that ∇^0 is a canonical connection. By Proposition 2.90, the canonical connection ∇^0 annihilates any K-invariant vector field. In particular, $\nabla^0 g = 0$. Hence $\nabla g = 0$ is equivalent to $\Lambda\mathfrak{m} \subset \mathfrak{so}(\mathfrak{m}, g_0)$. $\square$

Theorem 5.16. *Let g be a K-invariant metric on M, let $\iota : O^+M \to LM$ be the canonical morphism of principal bundles, and let $\xi : P(M, \mathrm{Spin}(\mathfrak{m}) \to O^+M$ be a Spin structure on M. The following sets are in bijection:*

(a) $\mathrm{Conn}_K(M, g)$,

(b) $\mathrm{Conn}_K(O^+M, M)$, *and*

(c) $\mathrm{Conn}_K(P(M, \mathrm{Spin}(\mathfrak{m})); M)$.

Proof. It follows from Proposition 2.37 (a) that the bijection between (a) and (b) is given by ι_*. The bijection between (b) and (c) is given by ξ_*, due to Proposition 2.37 (b). $\square$

Remark 5.17. We may also consider the correspondence in Theorem 5.16 at the level of connection maps. Let $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))^g_{\lambda_*}$ denote the set of $\Lambda \in \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))_{\lambda_*}$ such that $\Lambda\mathfrak{k} \subset \mathfrak{so}(\mathfrak{m}, g_o)$. Then Theorem 2.77 and Theorem 5.16 imply the bijection

$$\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))^g_{\lambda_*} \leftrightarrow \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{so}(\mathfrak{m}, g_o))_{\lambda_*}.$$

Notice that, in fact, $\lambda_* = \mathrm{ad}_{\mathfrak{h}}$ in this case.

5.3 DIRAC OPERATORS ASSOCIATED TO CARTAN-SCHOUTEN CONNECTIONS.

Let S be a connected compact simple Lie group, and let B denote its Killing form which is an $\mathrm{Ad}|_{\Delta(S\times S)}$-invariant, negative definite bilinear form on its Lie algebra $\mathfrak{s}$. We denote by $\mathrm{C}\ell(\mathfrak{s})$ the Clifford algebra associated to $(\mathfrak{s}, -B)$.

Definition 5.18. A Lie group S is said to be *spin* if the adjoint representation $\mathrm{Ad} : S \to SO(\mathfrak{s}, -B)$ lifts to the double cover $\xi : \mathrm{Spin}(\mathfrak{s}) \to SO(\mathfrak{s})$. In this case we denote by $\mathrm{Ad}^\xi : S \to \mathrm{Spin}(\mathfrak{s})$ the lift.

Remark 5.19. If S is simply-connected, then it is spin.

Definition 5.20. The *spin representation* of S is (σ, W) with $\sigma = \underline{\pi} \circ \mathrm{Ad}^\xi$, where $(\underline{\pi}, W)$ is the Spin representation of $\mathrm{Spin}(\mathfrak{s})$.

We equip S with the bi-invariant action (4.0.1) and view S as a reductive homogeneous $(S\times S)$-space, the $\mathrm{Ad}|_{\Delta(S\times S)}$-invariant decomposition $\mathfrak{s}\bigoplus\mathfrak{s} = \Delta(\mathfrak{s}\bigoplus\mathfrak{s})\bigoplus\mathfrak{m}_-$, where $\mathfrak{m}_- = \mathfrak{s}\bigoplus\{0\} \simeq \mathfrak{s}$.

To further simplify the analysis, we shall take advantage of the global trivialization $TS \simeq S\times\mathfrak{s}$ (Example 2.5) and work on the trivial bundle $S\times\mathfrak{s}\to S$ instead. This is legitimate, as the $S\times S$-action on S lifts to TS naturally, and it can be verified that the $S\times S$ action

$$(a,b)\cdot(s,Y) = (asb^{-1}, \mathrm{Ad}(b)Y) \tag{5.3.1}$$

makes the bundle isomorphism $TS\simeq S\times\mathfrak{s}$ an $S\times S$-equivariant isomorphism. Moreover, the inner product $g_o = -B$ extends to an $S\times S$-equivariant Riemannian metric on S, which we denote by g.

Proposition 5.21. *The Cartan-Schouten connections are metric connections.*

Proof. Since Cartan-Schouten connections are canonical connections and g is $S\times S$-invariant, it follows from Proposition 2.90 that $\nabla g = 0$. $\square$

Definition 5.22. The *Clifford bundle* on the Lie group S is defined as $S\times \mathrm{C}\ell(\mathfrak{s})$, the *spin structure* is $S\times\mathrm{Spin}(\mathfrak{s})$, and the *spinor bundle* is defined as $S\times W$.

Remark 5.23. The Clifford bundle $S\times\mathrm{C}\ell(\mathfrak{s})$ and the spinor bundle $S\times W$ are both $S\times S$-homogeneous bundles. More precisely, let $(a,b)\in S\times S$, $s\in S$, $\gamma\in\mathrm{C}\ell(\mathfrak{s})$, and $w\in W$. Then, $S\times S$ acts on $S\times\mathrm{C}\ell(\mathfrak{s})$ and $S\times W$, respectively, by

$$(a,b)\cdot(s,\gamma) = (asb^{-1}, c(\mathrm{Ad}(b))(\gamma)), \quad (a,b)\cdot(s,w) = (asb^{-1},\sigma(b)w).$$

We consider the spinor bundle $\mathfrak{S}_S = S\times W$ on S. The space of smooth sections with compact support is, due to the compactness of S, the space $C^\infty(S,W)$. We consider the normalized Haar measure $d\mu$ on S and the bundle metric

$$(\phi,\psi) = \int_S \langle\phi,\psi\rangle d\mu, \tag{5.3.2}$$

where $\phi,\psi\in C^\infty(S,W)$ ($\langle\cdot,\cdot\rangle$ is an S-invariant inner product on W).

Definition 5.24. The completion $L^2(S,W)$ of the space $C^\infty(S,W)$ with respect to the induced norm of the inner product (5.3.2) is called the *spinor space* on S.

We would like to discuss the actions of $C\ell(\mathfrak{s})$ and $S \times S$ on the spinor space $L^2(S, W)$. It is enough to identify the actions on the dense subspace $C^\infty(S, W)$.

Remark 5.25. If $E \xrightarrow{\pi} M$ is a K-invariant bundle and $\Phi : M \to E$ is a smooth section, and $k \in K$, then

$$(k \cdot \Phi)(m) := k \cdot \Phi(k^{-1} \cdot m)$$

is also a section. This defines an action of K on $\Gamma(M, E)$. If $E = M \times V \xrightarrow{\pi} M$ is a K-invariant trivial bundle then smooth section $\Phi : M \to E$ is of the form

$$\Phi(m) = (m, f(m)), \quad m \in M,$$

for some $f \in C^\infty(M, V)$. The above action of K on $\Gamma(M, E)$ translates into an action of K on $C^\infty(M, V)$.

Let us make precise the $S \times S$ action on $C^\infty(S, W)$ obtained as in the above Remark. Let $f \in C^\infty(S, W)$ and $(a, b) \in S \times S$, $x \in S$. Then

$$((a, b) \cdot f)(x) = \sigma(b)(f(axb^{-1})).$$

Definition 5.26. Let $f \in C^\infty(S, W)$, $a, b \in S$, $x \in S$. Define

$$\underline{\ell}(a)f(x) := f(a^{-1}x),$$
$$(\underline{r}(b)f)(x) := f(xb),$$
$$(\underline{\pi}_r(b)f)(x) := \sigma(b)f(xb).$$

Note that $\underline{\ell}$, $\underline{r}$, and $\underline{\pi}_r$ are representations of S on $C^\infty(S, W)$.

Definition 5.27. Let $f \in C^\infty(S, W)$, $\gamma \in C\ell(\mathfrak{s})$, $x \in S$. Define

$$(\underline{\pi}(\gamma)f)(x) := \underline{\pi}(\gamma)(f(x)).$$

Note that $\underline{\pi}$ is a representation of $C\ell(\mathfrak{s})$ on $C^\infty(S, W)$.

Proposition 5.28. *Let $\gamma \in C\ell(\mathfrak{s})$, $b \in S$. Then,*

(a) $\underline{r}(b)\underline{\pi}(\gamma) = \underline{\pi}(\gamma)\underline{r}(b)$.

(b) $\underline{\pi}_r(b)\underline{\pi}(\gamma) = \underline{\pi}(c(\mathrm{Ad}(b))(\gamma))\underline{\pi}_r(b)$.

Proof. Let $x \in S$ and $f \in C^\infty(S, W)$ arbitrarily. By definition,

$$(\underline{r}(b)\underline{\pi}(\gamma)f)(x) = \underline{r}(b)\underline{\pi}(\gamma)(f(x)) = \underline{\pi}(\gamma)f(xb) = \underline{\pi}(\gamma)\underline{r}(b)f(x),$$

which gives (a). To prove Part (b), observe that by the uniqueness in Corollary 5.3, we may write $c(\mathrm{Ad}(b))(\gamma) = \mathrm{Ad}^{\xi}(b)\cdot\gamma\cdot\mathrm{Ad}^{\xi}(b)^{-1}$, so that

$$\sigma(b)\underline{\pi}(\gamma) = \underline{\pi}(\mathrm{Ad}^{\xi}(b)\cdot\gamma) = \underline{\pi}(c(\mathrm{Ad}(b))(\gamma)\mathrm{Ad}^{\xi}(b)) = \underline{\pi}(c(\mathrm{Ad}(b))(\gamma))\sigma(b)$$

as $\sigma = \underline{\pi}\circ\mathrm{Ad}^{\xi}$. It then follows readily from the above identity that, indeed,

$$\underline{\pi}_r(b)\underline{\pi}(\gamma)f(x) = \sigma(b)\underline{\pi}(\gamma)f(xb) = \underline{\pi}(c(\mathrm{Ad}(b))(\gamma))\underline{\pi}_r(b)f(x).$$

$\square$

Notation. For the rest of this chapter we fix $\{X_i\}_{1\leq i\leq n}$ an orthonormal basis of $(\mathfrak{s},-B)$. Now $X_i^2=-1$ for all $1\leq i\leq n$.

Let $\iota_{\mathfrak{so}(\mathfrak{s})}:\mathfrak{so}(\mathfrak{s})\to \mathrm{C}\ell(\mathfrak{s})$ be the canonical inclusion induced by $\mathrm{Spin}(\mathfrak{s})\subset\mathrm{C}\ell^{\times}(\mathfrak{s})$. Consider the Lie algebra map

$$j_{\mathfrak{s}} = \iota_{\mathfrak{so}(\mathfrak{s})}\circ\mathrm{ad}_{\mathfrak{s}}:\mathfrak{s}\to\mathrm{C}\ell(\mathfrak{s}).$$

Consider also the natural inclusion map $\iota_{\mathfrak{s}}:\mathfrak{s}\to U(\mathfrak{s})$. Then the map

$$1\otimes\iota_{\mathfrak{s}} + j_{\mathfrak{s}}\otimes 1:\mathfrak{s}\to\mathrm{C}\ell(\mathfrak{s})\bigotimes U(\mathfrak{s})$$

is a Lie algebra homomorphism and therefore extends to an algebra map $U(\mathfrak{s})\to\mathrm{C}\ell(\mathfrak{s})\bigotimes U(\mathfrak{s})$, which we still denote by $1\otimes\iota_{\mathfrak{s}}+j_{\mathfrak{s}}\otimes 1$.

Proposition 5.29. *Let ∇^{α} be the covariant derivative corresponding to the Cartan-Schouten connection $\Gamma_{\alpha}\in\mathrm{Hom}_S(\mathfrak{s},\mathfrak{so}(\mathfrak{s}))$. Let $\widetilde{\nabla}^{\alpha}$ be the induced covariant derivative of ∇^{α}, acting on the space $C^{\infty}(S,W)$. Then for $X\in\mathfrak{m}_-$,*

$$\widetilde{\nabla}^{\alpha}_X = L_X + \frac{1-\alpha}{2}j_{\mathfrak{s}}(X).$$

Proof. By Theorem 2.87,

$$\underline{\pi}_*\circ\Lambda_{\alpha}X = \widetilde{\nabla}^{\alpha}_X - L_X$$

for $X\in\mathfrak{m}_-$. Proposition 4.6 shows that $\Lambda_{\alpha}=\frac{1-\alpha}{2}\mathrm{ad}_{\mathfrak{s}}$. The conclusion follows from the fact that $\mathrm{Ad}^{\xi}_*=\mathrm{ad}_{\mathfrak{s}}$. $\square$

Proposition 5.30. *Let $X \in \mathfrak{m}_-$, $\Phi \in \Gamma(S, S \times W)$, $\Phi(s) = (s, f(s))$ for $f \in C^\infty(S, W)$. Then $L_X\Phi(s) = (s, (\underline{r}_*(X)f)(s))$.*

Proof. By definition of Lie differentiation,

$$\begin{aligned} L_X\Phi(s) &= \frac{d}{dt}\Phi(s \cdot \exp tX)|_{t=0} = \frac{d}{dt}(s \cdot \exp tX, f(s \cdot \exp tX))|_{t=0} \\ &= (s, \frac{d}{dt}f(s \cdot \exp tX)|_{t=0}) = (s, (\underline{r}_*(X)f)(s)). \end{aligned}$$

□

The action of the Dirac operator D_α associated to $\widetilde{\nabla}_\alpha$ on $\Gamma(S, S \times W)$ induces an action of D_α on $C^\infty(S, W)$.

Proposition 5.31. *The action of D_α on $C^\infty(S, W)$ is given by $(\underline{\pi} \otimes \underline{r}_*)(\mathbb{D}_\alpha)$, where*

$$\mathbb{D}_\alpha = \sum_i (X_i \otimes 1)(1 \otimes X_i + \frac{1-\alpha}{2} j_{\mathfrak{s}}(X_i) \otimes 1) \in \mathrm{C}\ell(\mathfrak{s}) \bigotimes U(\mathfrak{s}).$$

Proof. We shall view each $X \in \mathfrak{s}$ as an element of the universal enveloping algebra $U(\mathfrak{s})$, or the Clifford algebra $\mathrm{C}\ell(\mathfrak{s})$, respectively. Note that $U(\mathfrak{s}) \simeq 1 \bigotimes U(\mathfrak{s})$ and $\mathrm{C}\ell(\mathfrak{s}) \simeq \mathrm{C}\ell(\mathfrak{s}) \bigotimes 1$. Then by linearity,

$$(\underline{\pi} \otimes \underline{r}_*)(\mathbb{D}_\alpha) = \sum_i (\underline{\pi} \otimes \underline{r}_*)((X_i \otimes 1)(1 \otimes X_i + \frac{1-\alpha}{2} j_{\mathfrak{s}}(X_i) \otimes 1)),$$

where the summand for each i is given by the product of

$$\begin{gathered} (\underline{\pi} \otimes \underline{r}_*)(X_i \otimes 1) = \underline{\pi}(X_i) \otimes 1, \\ (\underline{\pi} \otimes \underline{r}_*)(1 \otimes X_i + \frac{1-\alpha}{2} j_{\mathfrak{s}}(X_i) \otimes 1) = 1 \otimes \underline{r}_*(X_i) + \underline{\pi}(\frac{1-\alpha}{2} j_{\mathfrak{s}}(X_i) \otimes 1). \end{gathered}$$

It then follows from Proposition 5.29 and Proposition 5.30 that

$$(\underline{\pi} \otimes \underline{r}_*)(\mathbb{D}_\alpha) = \sum_i X_i \cdot (L_{X_i} + \frac{1-\alpha}{2} j_{\mathfrak{s}}(X_i)) = \sum_i X_i \cdot \widetilde{\nabla}^\alpha_{X_i} = D_\alpha.$$

□

Definition 5.32. The element

$$\Omega_{\mathfrak{s}} = -\sum_i X_i^2$$

is called the *Casimir element* of the Lie algebra $\mathfrak{s}$.

Remark 5.33. The Casimir element $\Omega_{\mathfrak{s}}$ is independent of the choice of the basis X_i, is a well-defined element in the universal enveloping algebra $U(\mathfrak{s})$, and belongs to the center of $U(\mathfrak{s})$. We remark that

$$(1 \otimes \iota_{\mathfrak{s}} + j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}}) \in U(\mathfrak{s}) \bigotimes \mathrm{C}\ell(\mathfrak{s}).$$

The main result on [22] can be restated as follows.

Theorem 5.34. [22] *The square of the Dirac operator $\mathbb{D}_\alpha$ may be represented in terms of the Casimir element $\Omega_{\mathfrak{s}}$ as*

$$\mathbb{D}_\alpha^2 = \frac{1}{2}(3\alpha+1)1 \otimes \Omega_{\mathfrak{s}} + \frac{1-3\alpha}{2}(1 \otimes \iota_{\mathfrak{s}} + j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}}) + \frac{1}{4}(3\alpha^2+1)j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1.$$

5.4 THE TWISTED DIRAC OPERATOR

As we have seen in Proposition 5.29, the action of $\widetilde{\nabla}_\alpha$ on $C^\infty(S, W)$ involves the Clifford algebra action of $j_{\mathfrak{s}}(X)$ and the Lie algebra action of L_X. In the geometric context studied above $j_{\mathfrak{s}}(X)$ acts as $\underline{\pi}(X)$ and L_X acts as $\underline{r}_*(X)$. Proposition 5.28(a) naturally leads us to consider the action of the Dirac operator D_α on $C^\infty(S, W)$ as the element $\mathbb{D}_\alpha \in U(\mathfrak{s}) \bigotimes \mathrm{C}\ell(\mathfrak{s})$.

However, there is a second natural Lie algebra action of $\mathfrak{s}$ on $L^2(S, W)$, $\underline{\pi}_{r,*}$, which is more interesting from the point of view of representation theory. In this situation Proposition 5.28(b) naturally leads us to consider the representation $\underline{\pi} \otimes \underline{\pi}_{r,*}$ of of the semi-direct product algebra $\mathrm{C}\ell(\mathfrak{s}) \rtimes U(\mathfrak{s})$ on $C^\infty(S, W)$. The purpose of this section is to compute the square of the Dirac operator as the element of $\mathrm{C}\ell(\mathfrak{s}) \rtimes U(\mathfrak{s})$.

Recall that $c(\mathrm{Ad}) : S \to \mathrm{Aut}_{\mathrm{alg}}(\mathrm{C}\ell(\mathfrak{s}))$ is a group morphism. This induces a Lie algebra morphism

$$c(\mathrm{Ad})_* : \mathfrak{s} \to \mathrm{End}(\mathrm{C}\ell(\mathfrak{s})).$$

The algebra $\mathrm{C}\ell(\mathfrak{s}) \rtimes U(\mathfrak{s})$ is defined as the vector space $\mathrm{C}\ell(\mathfrak{s}) \bigotimes U(\mathfrak{s})$ endowed with the algebra structure which make $1 \bigotimes U(\mathfrak{s})$ and $\mathrm{C}\ell(\mathfrak{s}) \bigotimes 1$ subalgebras, and for which

$$\begin{aligned}
&(X \otimes 1) \cdot (1 \otimes Y) = X \otimes Y, \\
&(1 \otimes Y) \cdot (X \otimes 1) = X \otimes Y + c(\mathrm{Ad})_*(Y)(X) \otimes 1
\end{aligned}$$

for all $X \in \mathrm{C}\ell(\mathfrak{s})$ and $Y \in \mathfrak{s} \subset U(\mathfrak{s})$.

Remark 5.35. No other relations are necessary: for both $U(\mathfrak{s})$ and $\mathrm{C}\ell(\mathfrak{s})$ are generated by $\mathfrak{s}$. Moreover, if $X, Y \in \mathfrak{s}$, then $c(\mathrm{Ad})_*(X)Y = [X, Y]$,

$$(1 \otimes Y) \cdot (X \otimes 1) = X \otimes Y + [Y, X] \otimes 1.$$

Definition 5.36. The *twisted Dirac operator* $\mathbf{D}_\alpha$ associated to the Cartan-Schouten connection Γ_α is given by

$$\mathbf{D}_\alpha = \sum_i (X_i \otimes 1)(1 \otimes X_i + \frac{1-\alpha}{2} j_{\mathfrak{s}}(X_i) \otimes 1) \in \mathrm{C}\ell(\mathfrak{s}) \rtimes U(\mathfrak{s}).$$

The algebra $\mathrm{C}\ell(\mathfrak{s}) \rtimes U(\mathfrak{s})$ is the non-commutative *Weil algebra* introduced in [**?**].

We will be interested in computing $\mathbf{D}_\alpha^2$, but first we will establish some useful facts.

5.4.1 Some identities.

For simplicity we denote $-B(\cdot,\cdot)$ by $\langle \cdot, \cdot \rangle$. Recall the composition of linear isomorphisms

$$\mu : \mathfrak{so}(\mathfrak{s}) \to \Lambda^2 \mathfrak{s} \quad \text{and} \quad \nu : \Lambda^2 \mathfrak{s} \to \mathrm{C}\ell^2(\mathfrak{s})$$

is $\iota_{\mathfrak{so}(\mathfrak{s})}$, the canonical isomorphism between $\mathfrak{so}(\mathfrak{s})$ and $\mathrm{C}\ell^2(\mathfrak{s})$ induced by $\mathrm{Spin}(\mathfrak{s}) \subset \mathrm{C}\ell^2(\mathfrak{s})$. More precisely,

$$\begin{aligned}
&\mu^{-1} : \Lambda^2 \mathfrak{s} \to \mathfrak{so}(\mathfrak{s});\ \mu^{-1}(X \wedge Y)(Z) = \langle X, Z \rangle Y - \langle Y, Z \rangle X, \\
&\nu : \Lambda^2 \mathfrak{s} \to \mathrm{C}\ell^2(\mathfrak{s});\ \ \mu(X \wedge Y) = \frac{1}{4}(X \cdot Y - Y \cdot X).
\end{aligned}$$

Lemma 5.37. *For each $X \in \mathfrak{s}$,*

$$\mu(\mathrm{ad}_{\mathfrak{s}}(X)) = \frac{1}{2}\sum_{s,t}\langle X, [X_s, X_t]\rangle X_s \wedge X_t$$

Proof. The following equation

$$\mu^{-1}(\sum_{s,t}\langle X, [X_s, X_t]\rangle X_s \wedge X_t = 2\mathrm{ad}_{\mathfrak{s}}(X)$$

holds when evaluated at each X_k, for $\mathrm{ad}_{\mathfrak{s}}(X)(X_k) = [X, X_k]$, and by the aforementioned formula for μ^{-1}, we have

$$\begin{aligned}\sum_{s,t}\langle X, [X_s, X_t]\rangle \mu^{-1}(X_s \wedge X_t)(X_k) &= \sum_{s,t}\langle X, [X_s, X_t]\rangle(\delta_{sk}X_t - \delta_{tk}X_s) \\ &= 2\sum_t \langle X, [X_s, X_t]\rangle X_t \\ &= 2\sum_t \langle [X, X_s], X_t\rangle X_t = 2\mathrm{ad}_{\mathfrak{s}}(X)(X_k).\end{aligned}$$

□

Proposition 5.38. *The Lie algebra map*

$$j_{\mathfrak{s}} : \mathfrak{s} \longrightarrow \mathrm{C}\ell(\mathfrak{s})$$

is given by

$$j_{\mathfrak{s}}(X) = \frac{1}{4}\sum_{s,t}\langle X, [X_s, X_t]\rangle X_s \cdot X_t \in \mathrm{C}\ell^2(\mathfrak{s}).$$

Proof. By definition of $j_{\mathfrak{s}}$ and Proposition 5.37,

$$\begin{aligned} j_{\mathfrak{s}}(X) &= \iota_{\mathfrak{so}(\mathfrak{s})} \circ \mathrm{ad}_{\mathfrak{s}}(X) = \nu \circ \mu(\mathrm{ad}_{\mathfrak{s}}(X)) \\ &= \frac{1}{2}\sum_{s,t}\langle X_a, [X_s, X_t]\rangle \nu(X_s \wedge X_t) \\ &= \frac{1}{8}\sum_{s,t}\langle X, [X_s, X_t]\rangle (X_s \cdot X_t - X_t \cdot X_s).\end{aligned}$$

Since $\langle X, [X_s, X_t]\rangle = -\langle X, [X_s, X_t]\rangle$, we conclude that

$$\langle X, [X_s, X_t]\rangle (X_s \cdot X_t - X_t \cdot X_s) = 2\langle X, [X_s, X_t]\rangle X_s \cdot X_t,$$

which yields the desired

$$\begin{aligned} j_{\mathfrak{s}}(X) &= \frac{1}{8}\sum_{s,t}\langle X, [X_s, X_t]\rangle (X_s \cdot X_t - X_t \cdot X_s) \\ &= \frac{1}{4}\sum_{s,t}\langle X, [X_s, X_t]\rangle X_s \cdot X_t. \end{aligned}$$

□

Lemma 5.39.

$$j_{\mathfrak{s}}(X_i) \cdot X_j = X_j \cdot j_{\mathfrak{s}}(X_i) + [X_i, X_j].$$

Proof. For each index s and t, element $X_s \cdot X_t \cdot X_j - X_j \cdot X_s \cdot X_t$ may be written as $(X_s \cdot X_t \cdot X_j + X_s \cdot X_j \cdot X_t) - (X_j \cdot X_s \cdot X_t + X_s \cdot X_j \cdot X_t) = 2\delta_{js} X_t - 2X_s \delta_{tj}$. Then by Proposition 5.38, we obtain

$$\begin{aligned} j_{\mathfrak{s}}(X_i) \cdot X_j - X_j \cdot j_{\mathfrak{s}}(X_i) &= \frac{1}{4}\sum_{s,t}\langle X_i, [X_s, X_t]\rangle 2(\delta_{js} X_t - X_s \delta_{tj}) \\ &= \sum_{s}\langle X_s, [X_i, X_j]\rangle X_s = [X_i, X_j]. \end{aligned}$$

□

Lemma 5.40. *Let $X, Y \in \mathfrak{s}$. Then*

$$c(\mathrm{Ad})_*(X)(j_{\mathfrak{s}}(Y)) = j_{\mathfrak{s}}([X, Y]).$$

Proof. By Proposition 5.38,

$$\begin{aligned} 4c(\mathrm{Ad})_*(X)(j_{\mathfrak{s}}(Y)) &= \sum_{s,t}\langle Y, [X_s, X_t]\rangle c(\mathrm{Ad})_*(X)(X_s \cdot X_t) \\ &= \sum_{s,t}\langle Y, [X_s, X_t]\rangle ([X, X_s] \cdot X_t + X_s \cdot [X, X_t]) \\ &= \sum_{s,t}\langle Y, [X_s, X_t]\rangle [X, X_s] \cdot X_t + \sum_{s,t}\langle Y, [X_s, X_t]\rangle X_s \cdot [X, X_t]. \end{aligned}$$

We rewrite this sum as

$$\sum_{s,t,a}\langle Y, [X_s, X_t]\rangle \langle [X, X_s], X_a\rangle \cdot X_a \cdot X_t + \sum_{s,t,b}\langle Y, [X_s, X_t]\rangle \cdot \langle [X, X_t], X_b\rangle X_s \cdot X_b.$$

After summing over a and b in a different manner, we conclude that, indeed,

$$\begin{aligned}
4c(\mathrm{Ad})_*(X)(j_{\mathfrak{s}}(Y)) &= \sum_{t,a}\langle [X_t, Y], [X_a, X]\rangle X_a \cdot X_t + \sum_{s,b}\langle [Y, X_s], [X_b, X]\rangle X_s \cdot X_b \\
&= \sum_{s,t}(\langle\langle [X_t, Y], [X_s, X]\rangle + \langle [Y, X_s], [X_t, X]\rangle X_s \cdot X_t \\
&= \sum_{s,t}(\langle\langle Y, [[X_s, X], X_t]\rangle + \langle Y, [X_s, [X_t, X]]\rangle X_s \cdot X_t \\
&= \sum_{s,t}\langle Y, [[X_s, X_t], X]\rangle X_s \cdot X_t \text{ (Jacobi identity)} \\
&= \sum_{s,t}\langle [X, Y], [X_s, X_t]\rangle X_s \cdot X_t = 4j_{\mathfrak{s}}([X, Y]).
\end{aligned}$$

□

Proposition 5.41. *The linear homomorphism*

$$1 \otimes \iota_{\mathfrak{s}} - j_{\mathfrak{s}} \otimes 1 : \mathfrak{s} \to \mathrm{C}\ell(\mathfrak{s}) \rtimes U(\mathfrak{s})$$

is a Lie algebra map.

Proof. For convenience we denote $1 \otimes \iota_{\mathfrak{s}} - j_{\mathfrak{s}} \otimes 1 =: k_{\mathfrak{s}}$. By Lemma 5.40,

$$\begin{aligned}
k_{\mathfrak{s}}(X)k_{\mathfrak{s}}(Y) &= (1 \otimes X - j_{\mathfrak{s}}(X) \otimes 1)(1 \otimes Y - j_{\mathfrak{s}}(Y) \otimes 1) \\
&= 1 \otimes XY - j_{\mathfrak{s}}(Y) \otimes X - j_{\mathfrak{s}}([X, Y]) \otimes 1 - j_{\mathfrak{s}}(X) \otimes Y + j_{\mathfrak{s}}(X)j_{\mathfrak{s}}(Y) \otimes 1,
\end{aligned}$$

and by symmetry

$$k_{\mathfrak{s}}(Y)k_{\mathfrak{s}}(X) = 1 \otimes YX - j_{\mathfrak{s}}(X) \otimes Y - j_{\mathfrak{s}}([Y, X]) \otimes 1 - j_{\mathfrak{s}}(Y) \otimes X + j_{\mathfrak{s}}(Y)j_{\mathfrak{s}}(X) \otimes 1$$

Since $j_{\mathfrak{s}}$ is a Lie algebra map, it follows that

$$k_{\mathfrak{s}}(X)k_{\mathfrak{s}}(Y) - k_{\mathfrak{s}}(Y)k_{\mathfrak{s}}(X) = 1 \otimes [X, Y] + j_{\mathfrak{s}}([X, Y]) \otimes 1 - 2j_{\mathfrak{s}}([X, Y]) \otimes 1 = k_{\mathfrak{s}}([X, Y]).$$

□

Remark 5.42. The Lie algebra map

$$1 \otimes \iota_{\mathfrak{s}} - j_{\mathfrak{s}} \otimes 1 : \mathfrak{s} \to \mathrm{C}\ell(\mathfrak{s}) \rtimes U(\mathfrak{s}) \tag{5.4.1}$$

extends to an algebra morphism $U(\mathfrak{s}) \to \mathrm{C}\ell(\mathfrak{s}) \rtimes U(\mathfrak{s})$ which we will still denote by $1 \otimes \iota_{\mathfrak{s}} - j_{\mathfrak{s}} \otimes 1$.

Lemma 5.43.

$$\sum_{i,j}[X_i, X_j] \cdot X_j \otimes X_i = -\sum_{i,j} X_i \cdot X_j \otimes [X_i, X_j].$$

$$\sum_{i,j} X_i \cdot [X_i, X_j] \otimes X_j = -\sum_{i,j} X_i \cdot X_j \otimes [X_i, X_j].$$

$$\sum_{i \neq j} X_i \cdot X_j \otimes X_i X_j = \frac{1}{2} \sum_{i,j} X_i \cdot X_j \otimes [X_i, X_j].$$

$$\sum_{i} j_{\mathfrak{s}}(X_i) \otimes X_i = \frac{1}{4} \sum_{i,j} X_i \cdot X_j \otimes [X_i, X_j].$$

Proof. These are all elementary manipulations of indices:

$$\begin{aligned}
\sum_{i,j}[X_i, X_j] \cdot X_j \otimes X_i &= \sum_{i,j,a} \langle X_a, [X_i, X_j] \rangle X_a \cdot X_j \otimes X_i \\
&= -\sum_{i,j,a} \langle X_i, [X_a, X_j] \rangle X_a \cdot X_j \otimes X_i \\
&= -\sum_{i,j,a} X_a \cdot X_j \otimes \langle X_i, [X_a, X_j] \rangle X_i \\
&= -\sum_{a,j} X_a \cdot X_j \otimes [X_a, X_j], \\
\sum_{i,j} X_i \cdot [X_i, X_j] \otimes X_j &= \sum_{i,j,b} X_i \cdot \langle [X_i, X_j], X_b \rangle X_b \otimes X_j \\
&= -\sum_{i,j,b} X_i \cdot X_b \otimes \langle [X_i, X_b], X_j \rangle X_j \\
&= -\sum_{i,b} X_i \cdot X_b \otimes [X_i, X_b], \\
\sum_{i \neq j} X_i \cdot X_j \otimes X_i X_j &= \sum_{i<j} X_i \cdot X_j \otimes (X_i X_j - X_j X_i) \\
&= \frac{1}{2} \sum_{i,j} X_i \cdot X_j \otimes [X_i, X_j], \\
\sum_{i,j} X_i \cdot X_j \otimes [X_i, X_j] &= \sum_{i,j,s} X_i \cdot X_j \otimes \langle [X_i, X_j], X_s \rangle X_s \\
&= \sum_{i,j,s} \langle [X_i, X_j], X_s \rangle X_i \cdot X_j \otimes X_s \\
&= 4 \sum_{s} j_{\mathfrak{s}}(X_s) \otimes X_s \text{ (Lemma 5.38)}.
\end{aligned}$$

□

Lemma 5.44.

$$\sum_{i,j} X_i \cdot X_j \otimes [X_i, X_j] = -2(1 \otimes \iota_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) + j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1 - (1 \otimes \iota_{\mathfrak{s}} - j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}})).$$

Proof. This follows from the observation that, by Proposition 5.41, we have

$$\begin{aligned}
(1 \otimes \iota_{\mathfrak{s}} - j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}}) &= -\sum_i (1 \otimes \iota_{\mathfrak{s}} - j_{\mathfrak{s}} \otimes 1)(X_i^2) \\
&= -\sum_i (1 \otimes X_i - j_{\mathfrak{s}}(X_i) \otimes 1)(1 \otimes X_i - j_{\mathfrak{s}}(X_i) \otimes 1) \\
&= -\sum_i (1 \otimes X_i^2 - 2j_{\mathfrak{s}}(X_i) \otimes X_i + j_{\mathfrak{s}}(X_i^2) \otimes 1) \\
&= 1 \otimes \iota_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) + 2\sum_i j_{\mathfrak{s}}(X_i) \otimes X_i + j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1,
\end{aligned}$$

which yields the conclusion due to the fourth identity in Lemma 5.43. □

Lemma 5.45.

$$\begin{aligned}
\sum_{i,j} X_i \cdot X_j \cdot j_{\mathfrak{s}}([X_i, X_j]) &= -4j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}). \\
\sum_{i,j} X_i \cdot [X_i, X_j] \cdot j_{\mathfrak{s}}(X_j) &= 4j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}).
\end{aligned}$$

Proof. The first identity follows from Lemma 5.38:

$$\begin{aligned}
j_{\mathfrak{s}}([X_i, X_j]) &= j_{\mathfrak{s}}(\sum_s \langle [X_i, X_j], X_s \rangle X_s) = \sum_s \langle [X_i, X_j], X_s \rangle j_{\mathfrak{s}}(X_s) \\
\sum_{i,j} X_i \cdot X_j \cdot j_{\mathfrak{s}}([X_i, X_j]) &= \sum_{i,j,s} X_i \cdot X_j \langle [X_i, X_j], X_s \rangle j_{\mathfrak{s}}(X_s) \\
&= 4\sum_k (j_{\mathfrak{s}}(X_k))^2 = -4j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}),
\end{aligned}$$

and so does the second identity:

$$\begin{aligned}
\sum_{i,j} X_i \cdot [X_i, X_j] \cdot j_{\mathfrak{s}}(X_j) &= \sum_{i,j,k} \langle [X_i, X_j], X_k \rangle X_i \cdot X_k \cdot j_{\mathfrak{s}}(X_j) \\
&= -\sum_{i,j,k} \langle X_j, [X_i, X_k] \rangle X_i \cdot X_k j_{\mathfrak{s}}(X_j) \\
&= -4\sum_j (j_{\mathfrak{s}}(X_j))^2 = 4j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}).
\end{aligned}$$

□

5.4.2 Main result.

Theorem 5.46. *The square of the twisted Dirac operator satisfies*

$$\mathbf{D}_\alpha^2 = \frac{7-3\alpha}{2}(1 \otimes \iota_{\mathfrak{s}}(\Omega_{\mathfrak{s}})) + \frac{3\alpha-5}{2}(1 \otimes \iota_{\mathfrak{s}} - j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}}) + \frac{3\alpha^2-12\alpha+13}{4} j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1.$$

Proof. For simplicity, let us denote $a = \frac{1-\alpha}{2}$. It follows from Definition 5.36 that

$$\begin{aligned}
\mathbf{D}_\alpha^2 &= \sum_{i,j}(X_i \otimes X_i + aX_i \cdot j_{\mathfrak{s}}(X_i) \otimes 1)(X_j \otimes X_j + aX_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1) \\
&= \sum_{i,j}(X_i \otimes X_i)(X_j \otimes X_j) \\
&+ a\sum_{i,j}(X_i \otimes X_i)(X_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1) + a\sum_{i,j}(X_i \cdot j_{\mathfrak{s}}(X_i) \otimes 1)(X_j \otimes X_j) \\
&+ a^2\sum_{i,j}(X_i \cdot j_{\mathfrak{s}}(X_i) \otimes 1)(X_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1). \qquad (5.4.2)
\end{aligned}$$

We analyze each term. By the first and the third identity of Lemma 5.43,

$$\begin{aligned}
\sum_{i,j}(X_i \otimes X_i)(X_j \otimes X_j) &= \sum_{i,j} X_i \cdot X_j \otimes X_iX_j + \sum_{i,j} X_i \cdot [X_i, X_j] \otimes X_j \\
&= -\sum_i 1 \otimes X_i^2 - \frac{1}{2}\sum_{i\neq j} X_i \cdot X_j \otimes [X_i, X_j].
\end{aligned}$$

It then follows from Lemma 5.44 that

$$\sum_{i,j}(X_i \otimes X_i)(X_j \otimes X_j) = 2(1 \otimes \iota_{\mathfrak{s}}(\Omega_{\mathfrak{s}})) + j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1 - (1 \otimes \iota_{\mathfrak{s}} - j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}}). \quad (5.4.3)$$

Before moving on to the next term, observe that $(X_i \otimes X_i)(X_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1)$ equals

$$X_i \cdot X_j \cdot j_{\mathfrak{s}}(X_j) \otimes X_i + X_i \cdot X_j \cdot j_{\mathfrak{s}}([X_i, X_j]) \otimes 1 + X_i \cdot [X_i, X_j] \cdot j_{\mathfrak{s}}(X_j) \otimes 1, \quad (5.4.4)$$

for it may be written as

$$\begin{aligned}
(X_i \otimes X_i)(X_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1) &= (X_i \otimes 1)(1 \otimes X_i)(X_j \otimes 1)(j_{\mathfrak{s}}(X_j) \otimes 1) \\
&= (X_i \otimes 1)(X_j \otimes X_i + [X_i, X_j] \otimes 1)(j_{\mathfrak{s}}(X_j) \otimes 1) \\
&= (X_i \otimes 1)(X_j \otimes X_i)(j_{\mathfrak{s}}(X_j) \otimes 1) \qquad (5.4.5) \\
&+ X_i \cdot j_{\mathfrak{s}}([X_i, X_j]) \cdot X_j \otimes 1,
\end{aligned}$$

where, using the following identity deduced easily from Lemma 5.40:

$$\begin{aligned}(1\otimes X_i)(j_{\mathfrak{s}}(X_j)\otimes 1) &= j_{\mathfrak{s}}(X_j)\otimes X_i + c(\mathrm{Ad})_*(X_i)(j_{\mathfrak{s}}(X_j))\\ &= j_{\mathfrak{s}}(X_j)\otimes X_i + j_{\mathfrak{s}}([X_i,X_j])\otimes 1,\end{aligned}$$

the term in (5.4.5) may be put into the desired

$$\begin{aligned}(X_i\otimes 1)(X_j\otimes X_i)(j_{\mathfrak{s}}(X_j)\otimes 1) &= (X_i\otimes 1)(X_j\otimes 1)(1\otimes X_i)(j_{\mathfrak{s}}(X_j)\otimes 1)\\ &= (X_i\otimes 1)(X_j\otimes 1)(j_{\mathfrak{s}}(X_j)\otimes X_i + j_{\mathfrak{s}}([X_i,X_j])\otimes 1)\\ &= X_i\cdot X_j\cdot j_{\mathfrak{s}}(X_j)\otimes X_i\\ &+ X_i\cdot X_j\cdot [X_i,X_j]\cdot j_{\mathfrak{s}}(X_j)\otimes 1.\end{aligned}$$

We also need the following identity, which follows easily from Lemma 5.39:

$$(X_i\cdot j_{\mathfrak{s}}(X_i)\otimes 1)(X_i\otimes X_i) = X_i\cdot j_{\mathfrak{s}}(X_i)\cdot X_i\otimes X_i = -j_{\mathfrak{s}}(X_i)\otimes X_i. \tag{5.4.6}$$

Now we are ready to compute the second term, using the expansion (5.4.4) and both identities in Lemma 5.45, as follows:

$$\begin{aligned}&\sum_{i,j}(X_i\otimes X_i)(X_j\cdot j_{\mathfrak{s}}(X_j)\otimes 1) \qquad (5.4.7)\\ &= \sum_i (X_i\otimes X_i)(X_i\cdot j_{\mathfrak{s}}(X_i)\otimes 1) + \sum_{i\neq j} X_i\cdot X_j\cdot j_{\mathfrak{s}}(X_j)\otimes X_i\\ &+ \sum_{i\neq j} X_i\cdot X_j\cdot j_{\mathfrak{s}}([X_i,X_j])\otimes 1 + \sum_{i\neq j} X_i\cdot [X_i,X_j]\cdot j_{\mathfrak{s}}(X_j)\otimes 1\\ &= \sum_i X_i^2\cdot j_{\mathfrak{s}}(X_i)\otimes X_i + \sum_{i\neq j} X_j\cdot X_i\cdot j_{\mathfrak{s}}(X_i)\otimes X_j - 4j_{\mathfrak{s}}(\Omega_{\mathfrak{s}})\otimes 1 + 4j_{\mathfrak{s}}(\Omega_{\mathfrak{s}})\otimes 1\\ &= -\sum_i j_{\mathfrak{s}}(X_i)\otimes X_i - \sum_{i\neq j} X_i\cdot X_j\cdot j_{\mathfrak{s}}(X_i)\otimes X_j.\end{aligned}$$

Next, we compute the third term, using (5.4.6):

$$\begin{aligned}&\sum_{i,j}(X_i\cdot j_{\mathfrak{s}}(X_i)\otimes 1)(X_j\otimes X_j) \qquad (5.4.8)\\ &= \sum_i (X_i\cdot j_{\mathfrak{s}}(X_i)\otimes 1)(X_i\otimes X_i) + \sum_{i\neq j} X_i\cdot j_{\mathfrak{s}}(X_i)\cdot X_j\otimes X_j\\ &= -\sum_i j_{\mathfrak{s}}(X_i)\otimes X_i + \sum_{i\neq j} X_i\cdot j_{\mathfrak{s}}(X_i)\cdot X_j\otimes X_j.\end{aligned}$$

Adding up the two terms (5.4.7) and (5.4.8) gives, by Lemma 5.43 and Lemma 5.44,

$$
\begin{aligned}
&\sum_{i,j}(X_i \otimes X_i)(X_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1) + \sum_{i,j}(X_i \cdot j_{\mathfrak{s}}(X_i) \otimes 1)(X_j \otimes X_j) \\
&= -2\sum_i j_{\mathfrak{s}}(X_i) \otimes X_i + \sum_{i\neq j} X_i(-X_j \cdot j_{\mathfrak{s}}(X_i) + j_{\mathfrak{s}}(X_i)\cdot X_j) \otimes X_j \\
&= -2\sum_i j_{\mathfrak{s}}(X_i) \otimes X_i + \sum_{i\neq j} X_i \cdot [X_i, X_j] \otimes X_j \text{ (using (5.39))} \\
&= -\frac{3}{2}\sum_{i,j} X_i \cdot X_j \otimes [X_i, X_j] \\
&= 3(1 \otimes \iota_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) + j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1 - (1 \otimes \iota_{\mathfrak{s}} - j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}})).
\end{aligned} \tag{5.4.9}
$$

Finally, using Lemma (5.39) and both identities in Lemma 5.44, the last term

$$
\begin{aligned}
&\sum_{i,j}(X_i \cdot j_{\mathfrak{s}}(X_i) \otimes 1)(X_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1) \\
&= \sum_i X_i^2 \cdot j_{\mathfrak{s}}(X_i^2) \otimes 1 + \sum_{i\neq j} X_i \cdot j_{\mathfrak{s}}(X_i) \cdot X_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1 \\
&= j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1 + \sum_{i\neq j} X_i \cdot [X_i, X_j] \cdot j_{\mathfrak{s}}(X_i) \otimes 1 + \sum_{i\neq j} X_i \cdot X_j \cdot j_{\mathfrak{s}}(X_i) \cdot j_{\mathfrak{s}}(X_j) \otimes 1 \\
&= 5 j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1 + \frac{1}{2}\sum_{i\neq j} X_i \cdot X_j \cdot j_{\mathfrak{s}}([X_i, X_j]) \otimes 1 = 3 j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1.
\end{aligned} \tag{5.4.10}
$$

We now substitute (5.4.3), (5.4.9), and (5.4.10) into (5.4.2), and the conclusion

$$
\mathbf{D}_\alpha^2 = (2+3a)(1 \otimes \iota_{\mathfrak{s}}(\Omega_{\mathfrak{s}})) - (3a+1)(1 \otimes \iota_{\mathfrak{s}} - j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}}) + (3a^2+3a+1) j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1
$$

follows as desired. □

Remark 5.47.

(1) Theorem 5.46 allows us investigate the so-called space of harmonic spinors, which is defined as the kernel of the action of $\mathbf{D}_\alpha$ on $L^2(S, W)$.

(2) The same argument would work and give an alternative proof of Theorem 5.34. The difference is that, as $\mathbb{D}_\alpha \in \mathrm{C}\ell(\mathfrak{s}) \bigotimes U(\mathfrak{s})$ was the Dirac operator studied in [22], the algebra map under consideration (the counterpart of (5.4.1) in [22]) was

$$
1 \otimes \iota_{\mathfrak{s}} + j_{\mathfrak{s}} \otimes 1 : U(\mathfrak{s}) \to \mathrm{C}\ell(\mathfrak{s}) \bigotimes U(\mathfrak{s}).
$$

We would need adjust Lemma 5.44 for that: in the space $\mathrm{C}\ell(\mathfrak{s})\bigotimes U(\mathfrak{s})$ it reads

$$\sum_{i,j} X_i \cdot X_j \otimes [X_i, X_j] = 2(1 \otimes \iota_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) + j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1 - (1 \otimes \iota_{\mathfrak{s}} + j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}})). \quad (5.4.11)$$

Indeed, the following identity still holds in $\mathrm{C}\ell(\mathfrak{s})\bigotimes U(\mathfrak{s})$:

$$\sum_{i,j} X_i \cdot X_j \otimes [X_i, X_j] = 4 \sum_s j_{\mathfrak{s}}(X_s) \otimes X_s.$$

(This is clear from its proof in Lemma 5.43.) Formula (5.4.11) then follows from that

$$\begin{aligned}
(1 \otimes \iota_{\mathfrak{s}} + j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}}) &= -\sum_i (1 \otimes \iota_{\mathfrak{s}} + j_{\mathfrak{s}} \otimes 1)(X_i^2) \\
&= -\sum_i (1 \otimes X_i + j_{\mathfrak{s}}(X_i) \otimes 1)(1 \otimes X_i + j_{\mathfrak{s}}(X_i) \otimes 1) \\
&= -\sum_i (1 \otimes X_i^2 + 2 j_{\mathfrak{s}}(X_i) \otimes X_i + j_{\mathfrak{s}}(X_i^2) \otimes 1) \\
&= 1 \otimes \iota_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) - 2 \sum_i j_{\mathfrak{s}}(X_i) \otimes X_i + j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1.
\end{aligned}$$

We now compute $\mathbb{D}_\alpha^2$. For simplicity, let us denote $a = \frac{1-\alpha}{2}$ (same as [22]). Then

$$\begin{aligned}
\mathbb{D}_\alpha^2 &= \sum_{i,j} (X_i \otimes X_i + a X_i \cdot j_{\mathfrak{s}}(X_i) \otimes 1)(X_j \otimes X_j + a X_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1) \\
&= \sum_{i,j} (X_i \otimes X_i)(X_j \otimes X_j) \\
&+ a \sum_{i,j} (X_i \otimes X_i)(X_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1) + a \sum_{i,j} (X_i \cdot j_{\mathfrak{s}}(X_i) \otimes 1)(X_j \otimes X_j) \\
&+ a^2 \sum_{i,j} (X_i \cdot j_{\mathfrak{s}}(X_i) \otimes 1)(X_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1).
\end{aligned}$$

We analyze each term. By the first and the third identity of Lemma 5.43,

$$\begin{aligned}
\sum_{i,j} (X_i \otimes X_i)(X_j \otimes X_j) &= \sum_i X_i^2 \otimes X_i^2 + \sum_{i \neq j} X_i \cdot X_j \otimes X_i \cdot X_j \\
&= -\sum_i 1 \otimes X_i^2 + \frac{1}{2} \sum_{i \neq j} X_i \cdot X_j \otimes [X_i, X_j] \\
&= 2(1 \otimes \iota_{\mathfrak{s}}(\Omega_{\mathfrak{s}})) - (1 \otimes \iota_{\mathfrak{s}} + j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}}) + j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1.
\end{aligned}$$

The second term and the third term may be combined into

$$
\begin{aligned}
&\sum_{i,j}(X_i \otimes X_i)(X_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1) + \sum_{i,j}(X_i \cdot j_{\mathfrak{s}}(X_i) \otimes 1)(X_j \otimes X_j) \\
&= \sum_i X_i^2 \cdot j_{\mathfrak{s}}(X_i) \otimes X_i + \sum_i X_i^2 \cdot j_{\mathfrak{s}}(X_i) \otimes X_i \\
&+ \sum_{i \neq j} X_i \cdot X_j \cdot j_{\mathfrak{s}}(X_j) \otimes X_i + \sum_{i \neq j} X_i \cdot j_{\mathfrak{s}}(X_i) \cdot X_j \otimes X_j \\
&= -2 \sum_i j_{\mathfrak{s}}(X_i) \otimes X_i - \sum_{i \neq j} X_i \cdot X_j \cdot j_{\mathfrak{s}}(X_i) \otimes X_j + \sum_{i \neq j} X_i \cdot j_{\mathfrak{s}}(X_i) \cdot X_j \otimes X_j \\
&= -2 \sum_i j_{\mathfrak{s}}(X_i) \otimes X_i + \sum_{i \neq j} X_i \cdot [X_i, X_j] \otimes X_j \text{ (Lemma 5.39)} \\
&= 3(1 \otimes \iota_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) + j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1 - (1 \otimes \iota_{\mathfrak{s}} + j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}})).
\end{aligned}
$$

Finally, it can be seen from the proof of Theorem 5.46 that the last term has the same expression as it would be in the space $\mathrm{C}\ell(\mathfrak{s}) \rtimes U(\mathfrak{s})$:

$$\sum_{i,j}(X_i \cdot j_{\mathfrak{s}}(X_i)) \otimes 1)(X_j \cdot j_{\mathfrak{s}}(X_j) \otimes 1) = 3 j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1.$$

We put these terms together, and the identity that appears in [22]

$$\mathbb{D}_\alpha^2 = (2 - 3a)(1 \otimes \iota_{\mathfrak{s}}(\Omega_{\mathfrak{s}})) + (3a - 1)(1 \otimes \iota_{\mathfrak{s}} + j_{\mathfrak{s}} \otimes 1)(\Omega_{\mathfrak{s}}) + (3a^2 - 3a + 1) j_{\mathfrak{s}}(\Omega_{\mathfrak{s}}) \otimes 1$$

follows as desired.

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BIBLIOGRAPHY

[1] I. Agricola, Connections on naturally reductive spaces, their Dirac operator and homogeneous models in string theory, *Comm. Math. Phys.* 232 (2003), no. 3, 535-563.

[2] A. Alekseev and E. Meinrenken, Lie theory and the Chern-Weil homomorphism, *Ann. Sci. École Norm. Sup.* 232 (2003), no. 3, 535-563.

[3] W. Ambrose and I. Singer, A theorem on holonomy, *Trans. Amer. Math. Soc.* 75 (1953).

[4] M. Atiyah and W. Schmid, A geometric construction of the discrete series for semisimple Lie groups, *Invent. Math.* 42 (1977), 1-62.

[5] A. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin Heidelberg, 1987.

[6] E. Cartan and J. Schouten, On the geometry of the group manifold of simple and semi-simple groups, *Proc. Amsterdam*, 29 (1926), 803-815.

[7] J. Duistermaat and J. Kolk, *Lie groups*, Universitext, Springer-Verlag, Berlin, 2000.

[8] C. Ehresmann, Les connexions infinitésimales dans un espace fibré différentiable, *Colloque de Topologie*, Bruxelles, 1950, 29-55.

[9] S. Goette, Equivariant η-invariants on homogeneous spaces, *Math. Z.*, 232 (1999), no. 1, 1-42.

[10] B. Gross, B. Kostant, P. Ramond, and S. Sternberg, The Weyl character formula, the half-spin representations, and equal rank subgroups, *Proc. Natl. Acad. Sci. USA* 95, (1998), no. 15, 8441-8442 (electronic).

[11] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, 34. American Mathematical Society, Providence, RI, 2001.

[12] A. Kai and K. Teh, Spectral action for a one-parameter family of Dirac operators on $SU(2)$ and $SU(3)$, *J. Math. Phys.* 54 (2013), no. 2, 022302.

[13] S. Kobayashi and K. Nomizu, *Foundation of Differential Geometry*, Wiley-Interscience, February 22, 1996.

[14] B. Kostant, A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups, *Duke Math. J.*, 100 (1999), no. 3, 447-501.

[15] B. Kostant, The Weyl algebra and the structure of all Lie superalgebras of Riemannian type, *Transform. Groups* 6 (2001), no. 3, 215-226.

[16] T. Laquer, Invariant affine connections on Lie groups, *Trans. Amer. Math. Soc.*, 331 (1992), no. 2, 541-551.

[17] B. Lawson and M.L. Michelsohn, *Spin geometry*, Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989.

[18] T. Levi-Civita, Nozione di parallelismo in una varietà qualunque, *Rend. Circ. Mat. Palermo* 42 (1917), 173-205.

[19] E. Meirenken, *Clifford algebras and Lie theory*, A Series of Modern Surveys in Mathematics, 58, Springer, Heidelberg, 2013.

[20] A. Oniščik, Decompositions of reductive Lie groups (Russian), *Mat. Sb. (N.S.)*, 80 (122) 1969, 553-599.

[21] R. Parthasarathy, Dirac operator and the discrete series, *Ann. of Math.*, (2) 96 (1972), 1-30.

[22] S. Slebarski, Dirac operators on Lie groups, *Bull. London Math. Soc.*, 17 (1985), no. 6, 579-583.

[23] S. Slebarski, The Dirac operator on homogeneous spaces and representations of reductive Lie groups. I., *Amer. J. Math.*, 109 (1987), no. 2, 283-301.

[24] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, January 1, 1999.